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Estimation and Goodness-of-Fit in the Case of

Randomly Censored Lifetime Data

DISSERTATION

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Abstract

Several parametric, nonparametric, and semi-parametric estimators of the distribution function of a randomly right-censored random variable are compared. A new continuous distribution function estimator for randomly censored data is developed, discussed, and compared to existing estimators. Minimum distance estimation is shown to be effective in estimating Weibull location parameters when random censoring is present. A method of estimating all 3 parameters of the 3-parameter Weibull distribution using a combination of minimum distance and maximum likelihood is also given. The mean integrated squared error is estimated for each estimator using Monte Carlo simulation and Kruskal-Wallis tests are used to discern which estimators are the best in the sense of having the smallest integrated squared error. A number of new goodness-of-fit tests for randomly censored data with a composite hypothesis are introduced. Cramér-von Mises and Anderson-Darling goodness-of-fit test statistics are modified to measure the discrepancy between the maximum likelihood estimate and the Kaplan-Meier product limit estimate of the distribution function of the random variable of interest. These modified test statistics are used to construct goodness-of-fit tests for the exponential, Weibull (shape 2), and Weibull (shape 3.5) distributions when the censoring distribution is assumed to be exponential. Percentage points are obtained via Monte Carlo simulation. Another test for the exponential with exponential censoring is constructed based on the knowledge that the minimum of exponentials is also exponential. More generally, elements of competing risks theory are used to build goodness-of-fit tests using crude lifetimes. One type of test requires a parametric fit to the crude lifetimes of both the variable of interest and the censoring random variable while the other type relies on the empirical survivor function of the crude lifetime of the censoring variable. In either case the assumption of an exponentially distributed censoring variable and special estimation techniques are no longer required, bringing much more flexibility to goodness-of-fit testing when samples are randomly right-censored. The powers of the

new KME-modified Cramér-von Mises and Anderson-Darling goodness-of-fit tests as well as the new tests based on crude lifetimes are compared to each other and to existing tests by Burke and Chen in the case of the test for exponentiality. Examples of crude life tests are presented and discussed.

Estimation and Goodness-of-Fit in the Case of Randomly Censored Lifetime Data

I. Background and Problem Statement

1.1 Background

The time until the occurrence of an event is randomly censored on the right when the observed portion of the time is arbitrarily truncated before the event, or failure, can be observed. This situation can be modeled using two random variables: the variable of interest and a censoring variable. The random variable of interest is randomly censored on the right by the censoring variable when only the minimum of the two can be observed. This is a competing risks model with two risks. The observer is usually able to distinguish failure times from censoring times. We will consider only the scenario where failure times are distinguishable from censoring times and restrict our attention to continuous univariate probability distributions throughout the dissertation.

The objective is to fit a random sample of the failure times with a parametric model. Random censorship makes this task more difficult because the actual failure time is not known when an item's lifetime is censored. What *is* known is that the censored observation survived *at least* until the time of censoring. To disregard the censored items and use only the observed failures would not provide an accurate representation of the true underlying process. Furthermore, the information provided by the knowledge that the suspended or censored item *survived* for a certain time period would be lost.

Some of the goodness-of-fit procedures for randomly censored data when the null distribution is completely specified require the assumption that the censoring distribution has a hazard rate that is proportional to that of the failure distribution. This proportional hazards model of random censorship has been explored by Koziol and Green [81], Koziol [80], and Csörgő and Horváth [31] among others. Many authors refer to this as the Koziol-Green model of random censorship. Distri-

bution theory for many goodness-of-fit statistics is very difficult, possibly intractable, particularly when parameters are estimated from data and especially when data is randomly censored. As a result, tables of percentage points are not available for testing goodness of fit. The development of a goodness-of-fit test can be approached from one of two directions: either (1) find an appropriate test statistic, preferably one with high power against a wide variety of alternatives, and attempt to derive the distribution of percentage points, or (2) construct a test statistic in such a way that it follows a known distribution for which percentage points are readily available.

The purpose of performing a goodness-of-fit test is to fit a random sample that is assumed to be independent and identically distributed with a parametric model in order to facilitate the characterization of the random variable as well as any desired statistical inference, reliability analysis, or maintenance planning. Quadratic distance measures such as the Cramér-von Mises and Anderson-Darling statistics provide a reasonable measure of the discrepancy between an empirical distribution function (EDF) and the distribution function of some parametric family with either known or estimated parameters and generally have superior power [123] among goodness-of-fit statistics for testing with uncensored samples. See D'Agostino and Stephens [32] for detailed descriptions of these statistics and more. In the case of a simple hypothesis where parameters are assumed known in the null hypothesis, the test statistic will measure the discrepancy between the EDF and a hypothetically true distribution function. However, in a goodness-of-fit test with a composite hypothesis in which parameters are estimated from a random sample, the only thing that is assumed to be true in the null hypothesis is the distributional family that characterizes the random variable of interest. EDF test statistics measure the discrepancy between the EDF, a nonparametric estimate, and a parametrically estimated distribution function usually obtained by maximum likelihood. This discrepancy is likely to be smaller when parameters are estimated from the data because both estimates are dependent on the same random sample. Although parameters are estimated from the data, the purpose of the goodness-of-fit test is still likely to be satisfied be-

cause the test verifies (or rules out) a parametric family of distributions that will (not) adequately characterize the random variable of interest.

1.2 Problem Statement

Data that is randomly censored complicates the estimation procedures and alters the distribution, and hence the percentage points, of goodness-of-fit test statistics. Furthermore, the EDF is no longer available as an estimator of the distribution function under random censorship. Researchers throughout the years have developed both parametric and nonparametric estimation techniques that will accommodate density and distribution function estimation for randomly censored samples. These estimators can be used to build goodness-of-fit tests. The effectiveness of an estimator is often measured in terms of mean integrated squared error (MISE). The smaller the MISE, the better the estimator. MISE is dependent on:

1. sample size;
2. the proportion of censoring;
3. the underlying lifetime distribution;
4. the method of estimation;
5. and possibly the distribution of the censoring random variable.

We would like to find an estimator with the smallest MISE for a wide variety of lifetime distributions. The problem may be formulated as follows. Let $\mathcal{M}$ denote our measure of MISE, let $\mathcal{E}$ be a class of distribution function estimators that can be used with randomly censored data, and let $\mathcal{F}$ represent a set of distribution function families. Given $F \in \mathcal{F}$ find $\hat{F} \in \mathcal{E}$ that minimizes $\mathcal{M}(\hat{F}, F)$.

Another method for comparing the relative sizes of values of populations is through the Kruskal-Wallis test. The Kruskal-Wallis test is a nonparametric test and, as such, requires no assumptions on the underlying populations being compared. Thus, when comparing skewed popu-

lations such as integrated squared error (ISE), the Kruskal-Wallis test should augment the results seen in the MISE comparison and by shedding light on whether the ISE values, in general, for one estimator are significantly lower than another. Therefore, the Kruskal-Wallis test will play a role in determining which estimators are better than others for the scenarios under consideration.

One particular problem in goodness-of-fit testing lies in the precise characterization of the distribution of the goodness-of-fit statistics. The distribution of a goodness-of-fit statistic must be known or estimated in order to know when to reject a given null hypothesis at a given level of significance when the value of a test statistic is calculated from observed data. While not necessary, it is very desirable for a goodness-of-fit test statistic to be invariant to location and scale changes in the distribution being tested. The distribution of the test statistic will be dependent on the censoring process as well as the amount of censoring in addition to the distributional family being tested and any shape parameters that may be involved.

The primary concern here within the context of the goodness-of-fit problem is with the composite hypothesis, where parameters must be estimated in order to test fit. With randomly censored lifetime data, the power of a goodness-of-fit test for a composite null hypothesis at a given significance level for a specified alternative distribution is dependent on:

1. sample size;
2. the proportion of censoring;
3. the family selected for the failure distribution;
4. the true underlying alternative distribution;
5. the censoring distribution;
6. the method of parameter estimation;
7. the method of nonparametric distribution function estimation;
8. the test statistic.

Items 2, 5, and 7 arise in the presence of random censoring. Item 7 vanishes in the case of goodness-of-fit tests using crude lifetimes, which are given in Sections 3.7 and 3.8. Power is defined as the probability of correctly rejecting the null hypothesis when, in reality, it is false. Our objective is to construct a test statistic that has the highest power among all other test statistics for the widest variety of alternatives under a given set of circumstances. The goodness-of-fit problem may be stated as follows. Let $\mathcal{T}$ be a set of goodness-of-fit test statistics that can be used with randomly censored data, $\mathcal{F}$ be a set of hypothesized distribution function families, and $\mathcal{G}$ be a set of alternative distribution function families. Further, let $\mathcal{P}$ denote the power of a given test statistic $T \in \mathcal{T}$ which is used to detect an alternative distribution $G \in \mathcal{G}$ when the hypothesized distribution is $F \in \mathcal{F}$. Given $F \in \mathcal{F}$ and $G \in \mathcal{G}$, find $T \in \mathcal{T}$ that maximizes $\mathcal{P}(T, F, G)$.

In addition to deriving a goodness-of-fit test with maximum power in a variety of settings, it is also important to consider flexibility. Consequently, we seek to maximize the flexibility of a goodness-of-fit test by avoiding restrictive assumptions regarding the distribution of the censoring variable. It is in this aspect that tests based on the fit of crude lifetimes stand alone above other tests in the case of randomly censored data. Until now the problem of testing goodness-of-fit with randomly censored data has always been done from the perspective of competing net lifetimes and has not been addressed as a mixture of crude lifetimes. The concepts of net and crude lifetimes are defined and discussed in Sections 3.7 and 3.8.

The problem of random right-censoring is of interest to the Air Force. One example of the need to overcome this problem is the characterization of the lifetime distribution of the Advanced Medium Range Air-to-Air Missile (AMRAAM) and its components. The AMRAAM is an active radar-guided missile carried by USAF F-15 and F-16 fighters. As the missiles accumulate captive carry hours the rigors of take-off, flight, and landing, as well as other environmental stresses can lead to a state of missile failure in which the missile is no longer capable of being launched. However, if a missile is successfully launched, permanently removed from the field for any reason, then it is

known to have survived until that time and is randomly censored because it is no longer observable until failure.

1.3 The Competing Risks Model of Random Censoring

The most widely used model in the literature to represent random right-censoring is that of two independent competing risks. The following notation will be used throughout this work. Let T be the random variable of interest with failure distribution function $F_T(t)$, survivor function $S_T(t) = 1 - F_T(t)$, and assume the density function $f_T(t) = F_T'(t)$ exists. Further, let C be a censoring random variable with distribution function $F_C(c)$, survivor function $S_C(c) = 1 - F_C(c)$, and assume the density function $f_C(c) = F_C'(c)$ exists. The expected proportion of failures is given by

$$\begin{aligned} p &= P[T \leq C] \\ &= \int_{-\infty}^{\infty} f_T(t) S_C(t) dt. \end{aligned}$$

The observed random sample consists of $x_i = \min\{t_i, c_i\}, i = 1, \ldots, n$, and the indicator $\delta_i = I_{[t_i \leq c_i]}$. Throughout this dissertation, the notation $t_{(i)}, i = 1, \ldots, r$, will be used to denote the set of ordered failure times, $c_{(i)}, i = 1, \ldots, n - r$, will denote the set of ordered withdrawal times, and $x_{(i)}, i = 1, \ldots, n$, will denote the entire set of ordered observations, failed or suspended. Additionally, r will be used to represent the sum $\sum_{i=1}^{n} \delta_i$, the number of observed failures in the sample.

In competing risks theory, a *net lifetime* represents the lifetime of an item when it is subject to one of the specified risks while no other risks are present and a *crude lifetime* represents the lifetime of an item subject to one of the specified risks when all risks are present [87: p. 110]. Our objective is to characterize the distribution of the net lifetime for the risk of interest, which is our underlying failure process. As competing risks, T and C in this context will be used to represent the net lifetimes and Y_T and Y_C will denote the crude lives that correspond to the net lifetimes T and

C. In other words, T represents the lifetime of the random variable of interest when no censoring is present while Y_T represents the lifetime of interest *in the presence of random censoring.* When the net lifetimes are independent, the relationship between survivor functions of X, T, C, Y_T, and Y_C is

$$S_X(t) = S_T(t)S_C(t) = pS_{Y_T}(t) + (1-p)S_{Y_C}(t).$$

A proof of this result is given in Section 3.7.

1.4 Organization of the Dissertation

The effectiveness of minimum distance estimation of Weibull location parameters for randomly censored samples is demonstrated using the Cramér-von Mises and Anderson-Darling statistics in Section 2.1.2. A technique for estimating all three parameters of a 3-parameter Weibull distribution which utilizes both minimum distance and maximum likelihood methods is outlined and demonstrated in Section 2.1.3. In Section 2.2 four nonparametric distribution function estimators are presented and a new continuous estimator of the distribution function is introduced. The new estimator is an extension of AFIT graduate James Sweeder's estimator [131] to randomly censored data, an idea brought forth by Dr. Albert Moore. The estimator is a trigonometrically smoothed and jackknifed Kaplan-Meier estimator. Two semi-parametric distribution function estimators are also presented and all of the estimators are compared.

Chapter III addresses goodness-of-fit tests in the case of randomly censored lifetime data. Asymptotic results for a class of goodness-of-fit statistics based on the Kaplan-Meier estimator are summarized and examined for the composite hypothesis case when the lifetime and censoring distributions are both exponential as well as the Weibull distribution within the proportional hazards model of censorship. Section 3.3 provides some justification for the use of the exponential distribution to model the random censoring variable. Eight new goodness-of-fit tests are introduced, including four new tests for exponentiality when the censoring variable is exponential and four new

tests for the Weibull distribution for shape parameters 2 and 3.5 when the censoring variable is exponential. Moreover, two promising new types of goodness-of-fit procedures for use when samples are randomly censored on the right are introduced, one using a simultaneous fit of crude lifetimes and one using a fit to the crude lifetime of the random variable of interest only. The second one may be classified as a partially parametric test of fit because the crude lifetime is fit with a parametric family while the resulting net lifetime depends on the EDF of the crude lifetime of the censoring random variable. This approach to goodness-of-fit testing with randomly censored data provides a tremendous increase in the applicability and flexibility of the tests because it requires no special estimation techniques, no special test statistics, and no special assumptions since complete sample goodness-of-fit tests are used. The only necessary assumption is that the net lifetimes of the failure and censoring distributions are independent. Although these new types of goodness-of-fit procedures focus on tests of composite hypothesis, many new tests for simple or composite hypotheses may be constructed similarly.

Percentage points for the new goodness-of-fit tests have been generated via Monte Carlo methods and are tabled in Sections 3.5 and 3.6. Section 3.9 contains power studies for all of the new tests for the exponential and Weibull distributions against several alternative distributions. The new tests for the exponential are compared to existing tests introduced by Burke [15] and C. H. Chen [21] in terms of power against the selected alternatives. Results and observations are summarized in Chapter IV.

II. Estimation

2.1 Parametric Estimation

2.1.1 Maximum Likelihood using the Censored-Data Likelihood Function. If we assume that the distributional family of a random variable is known, we can then estimate the parameters of that family using information from a random sample of observed values of the variable, even if they have been randomly censored. For distributions in which the survivor function has a closed form, the censored data maximum likelihood function can usually be used to estimate parameters without too much difficulty. The usual procedure in doing so, as in the complete sample case, is to maximize the likelihood function or its natural logarithm with respect to the unknown parameters of the distribution. That is, find the values of the parameters for which the likelihood function is maximized.

For an observed random sample $x_i = \min\{t_i, c_i\}, i = 1, \ldots, n$, with the indicator $\delta_i = I_{[t_i \le c_i]}$ in the competing risks model of random censorship, the likelihood function can be expressed as

$$L(x, \delta; \theta) = \prod_{i=1}^{n} [f_T(x_i, \theta) \cdot S_C(x_i)]^{\delta_i} [f_C(x_i) \cdot S_T(x_i, \theta)]^{1-\delta_i}$$

where θ is a vector of parameters. However, since we are not interested in estimating the parameters of the censoring distribution, the censored-data likelihood function can effectively be written as

$$L(x, \delta; \theta) = \prod_{i=1}^{n} f_T(x_i, \theta)^{\delta_i} S_T(x_i, \theta)^{1-\delta_i}.$$

In cases where the survivor function $S_T(x)$ does not have a closed form expression the complete sample likelihood function is appreciably simpler than the censored-data likelihood function, such as with the normal, lognormal, and gamma distributions. For example, the EM (Expectation-Maximization) algorithm of Dempster, Rubin, and Laird [34] may be used to find maximum likelihood estimates of parameters [28].

Figures 13, 14, and 15 in Appendix C show examples of maximum likelihood estimates of the distribution functions of the exponential, Weibull (shape 2), and Weibull (shape 3.5) distributions. The choice to work with these distributions stems from their wide applicability in lifetime data analysis and the variety in density shape. The exponential distribution is heavily skewed to the right while the Weibull with shape 3.5 is nearly symmetric. The skewness of the Weibull with shape 2 is roughly in between. Examples of the effects of varying levels of random censoring can be seen in the figures in Appendix C that have been generated for each estimation technique. A typical pattern that emerges is that the heavier the censoring the more the distribution function tends to be overestimated, thus the survivor function is underestimated. Please note, however, in Figure 15 that although the best estimate in the plot occurred when there was 75% expected censoring, this is generally not the case.

2.1.2 New Minimum Distance Methods for Randomly Censored Data. In this section, we extend minimum distance estimation in such a way as to include the estimation of parameters in the case of randomly censored samples. Minimum distance estimation, introduced by Wolfowitz [148, 149], has been demonstrated by Parr and Schucany [111], Hobbs, Moore, and Miller [64], and Gallagher and Moore [52] to be a robust estimation technique for complete samples, particularly when estimating location parameters. For the 3-parameter Weibull distribution, Gallagher and Moore [52] show for complete samples that using minimum distance to estimate the location parameter and maximum likelihood for shape and scale parameters produced the best overall results. In comparing several methods involving distance measures, they [52] further show for the Weibull distribution that the Anderson-Darling goodness-of-fit statistic is the preferred distance measure with the Cramér-von Mises statistic not far behind. As a result, minimum distance estimation using the Anderson-Darling and Cramér-von Mises statistics may be used to estimate parameters for randomly censored samples by replacing the EDF with the Kaplan-Meier product-limit estimator, which is detailed in Section 2.2.1. The Kaplan-Meier estimator (KME) is relatively

easy to compute, it can be integrated analytically, and it reduces to the EDF in the case of no censoring.

For complete samples under a simple hypothesis, the Cramér-von Mises, W^2, and Anderson-Darling, A^2, statistics are defined as

$$W^2 = n \int_{-\infty}^{+\infty} [F_n(x) - F_0(x)]^2 dF_0(x) \tag{1}$$

and

$$A^2 = n \int_{-\infty}^{+\infty} \frac{[F_n(x) - F_0(x)]^2}{F_0(x)[1 - F_0(x)]} dF_0(x) \tag{2}$$

where $F_n(x)$ is the EDF and $F_0(x)$ is the distribution of the random variable X. Computing formulas are given in [32: page 101]. The EDF $F_n(x)$, however, is not available for randomly censored samples and may be replaced by a suitable estimator of the distribution function, such as the Kaplan-Meier. The modified test statistics become

$$W_{r,n}^2 = n \int_{-\infty}^{+\infty} [\hat{F}_n(x) - F_0(x)]^2 dF_0(x)$$

and

$$A_{r,n}^2 = n \int_{-\infty}^{+\infty} \frac{[\hat{F}_n(x) - F_0(x)]^2}{F_0(x)[1 - F_0(x)]} dF_0(x).$$

Upon making this substitution, new computing formulas must be derived. Making use of the probability integral transform, $U = F_0(X)$, and letting $t_{(j)}, 1 \leq j \leq r$, denote the ordered uncensored observations where r is the number of failures in the set, we obtain the transformed set of orderd sample values $U_{(j)} = F_0(t_{(j)})$. It will also be necessary to let $t_{(0)} = -\infty$ and $t_{(r+1)} = +\infty$, hence $U_{(0)} = 0$ and $U_{(r+1)} = 1$. Beginning with the modified Cramér-von Mises statistic

$$W_{r,n}^2 = n \int_{-\infty}^{+\infty} [\hat{F}_n(x) - F_0(x)]^2 dF_0(x)$$

we make use of the probability integral transform, the fact that $\hat{F}_n(x) - F_0(x) = \hat{F}_n(Q_0(u)) - u$, where $Q_0(u) = F_0^{-1}(u) = X$, assuming $F_0^{-1}(u)$ exists, and express the integral as the sum of integrals over the ordered set of failures to get

$$W_{r,n}^2 = n \sum_{j=1}^{r+1} \int_{U_{(j-1)}}^{U_{(j)}} [\hat{F}_n(Q_0(u)) - u]^2 du.$$

Expanding the binomial and integrating leads to

$$W_{r,n}^2 = n \sum_{j=1}^{r+1} [\hat{F}_n(x_{(j-1)})]^2 (U_{(j)} - U_{(j-1)}) - \hat{F}_n(x_{(j-1)})(U_{(j)}^2 - U_{(j-1)}^2) + \frac{1}{3}(U_{(j)}^3 - U_{(j-1)}^3)$$

since $\hat{F}_n(Q_0(u)) = \hat{F}_n(Q_0(U_{(j-1)})) = \hat{F}_n(t_{(j-1)})$ is a constant within each interval $U_{(j-1)}$ to $U_{(j)}$. Simplifying and noting that $\hat{F}_n(t_{(0)}) = 0$ gives

$$W_{r,n}^2 = n \sum_{j=2}^{r+1} [\hat{F}_n(x_{(j-1)})]^2 (U_{(j)} - U_{(j-1)}) - \hat{F}_n(x_{(j-1)})(U_{(j)}^2 - U_{(j-1)}^2) + \frac{n}{3}$$

which can then be written as

$$W_{r,n}^2 = n \sum_{j=1}^{r} [\hat{F}_n(x_{(j)})]^2 (U_{(j+1)} - U_{(j)}) - \hat{F}_n(x_{(j)})(U_{(j+1)}^2 - U_{(j)}^2) + \frac{n}{3}. \quad (3)$$

The computing formula for the modified Anderson-Darling statistic is obtained similarly as follows. Beginning with

$$A_{r,n}^2 = n \int_{-\infty}^{+\infty} \frac{[\hat{F}_n(x) - F_0(x)]^2}{F_0(x)[1 - F_0(x)]} dF_0(x)$$

we make the same substitutions as with W^2 above and express the integral as the sum of integrals over the ordered failure set to get

$$A_{r,n}^2 = n \sum_{j=1}^{r+1} \int_{U_{(j-1)}}^{U_{(j)}} \frac{[\hat{F}_n(Q_0(u)) - u]^2}{u(1-u)} du.$$

Expanding the numerator yields

$$A_{r,n}^2 = n\sum_{j=1}^{r+1}\int_{U_{(j-1)}}^{U_{(j)}}\left\{\frac{\hat{F}_n(Q_0(u))^2 - 2u\hat{F}_n(Q_0(u)) + u^2}{u(1-u)}\right\}du$$

which can be written as

$$A_{r,n}^2 = n\sum_{j=1}^{r+1}\int_{U_{(j-1)}}^{U_{(j)}}\left\{\frac{\hat{F}_n(Q_0(u))^2}{u} + \frac{\hat{F}_n(Q_0(u))^2}{1-u} - \frac{2\hat{F}_n(Q_0(u))}{1-u} + \frac{u}{1-u}\right\}du.$$

Completing the square in

$$A_{r,n}^2 = n\sum_{j=1}^{r+1}\int_{U_{(j-1)}}^{U_{(j)}}\left\{\frac{\hat{F}_n(Q_0(u))^2}{u} + \frac{\hat{F}_n(Q_0(u))^2}{1-u} - \frac{2\hat{F}_n(Q_0(u))}{1-u} + \frac{1}{1-u} - \frac{1}{1-u} + \frac{u}{1-u}\right\}du$$

gives

$$A_{r,n}^2 = n\sum_{j=1}^{r+1}\int_{U_{(j-1)}}^{U_{(j)}}\left\{\frac{\hat{F}_n(Q_0(u))^2}{u} + \frac{[\hat{F}_n(Q_0(u)) - 1]^2}{1-u} - 1\right\}du.$$

Again, noting that $\hat{F}_n(Q_0(u)) = \hat{F}_n(Q_0(U_{(j-1)})) = \hat{F}_n(x_{(j-1)})$ and is a constant within each interval $U_{(j-1)}$ to $U_{(j)}$, the integration gives

$$A_{r,n}^2 = n\sum_{j=1}^{r+1}\{[\hat{F}_n(x_{j-1})]^2\log(u)|_{U_{(j-1)}}^{U_{(j)}} - [\hat{F}_n(x_{(j-1)} - 1]^2\log(1-u)|_{U_{(j-1)}}^{U_{(j)}} - u|_{U_{(j-1)}}^{U_{(j)}}\}. \tag{4}$$

Simplifying and distributing in Equation 4 yields

$$\begin{aligned} A_{r,n}^2 \quad = \quad & -n + n\{\textstyle\sum_{j=1}^{r+1}[\hat{F}_n(x_{j-1})]^2\log(U_{(j)}) - \sum_{j=1}^{r+1}[\hat{F}_n(x_{j-1})]^2\log(U_{(j-1)}) \\ & - \textstyle\sum_{j=1}^{r+1}[\hat{F}_n(x_{j-1}) - 1]^2\log(1-U_{(j)}) + \sum_{j=1}^{r+1}[\hat{F}_n(x_{j-1}) - 1]^2\log(1-U_{(j-1)})\}. \end{aligned} \tag{5}$$

Now, since $\log(U_{(r+1)}) = 0, \hat{F}_n(x_{(0)}) = 0, \hat{F}_n(x_{(r+1)}) - 1 = 0$, and $\log(1 - u_{(0)}) = 0$, Equation 5 can be written as

$$\begin{aligned} A^2_{r,n} &= -n + n\{\sum_{j=1}^{r}[\hat{F}_n(x_{j-1})]^2 \log(U_{(j)}) - \sum_{j=2}^{r+1}[\hat{F}_n(x_{j-1})]^2 \log(U_{(j-1)}) \\ &\quad - \sum_{j=1}^{r}[\hat{F}_n(x_{j-1}) - 1]^2 \log(1 - U_{(j)}) + \sum_{j=2}^{r+1}[\hat{F}_n(x_{j-1}) - 1]^2 \log(1 - U_{(j-1)})\} \end{aligned}$$

which simplifies to

$$\begin{aligned} A^2_{r,n} &= -n + n\{\sum_{j=1}^{r}[\hat{F}_n(x_{j-1})]^2 \log(U_{(j)}) - \sum_{j=1}^{r}[\hat{F}_n(x_j)]^2 \log(U_{(j)}) \\ &\quad - \sum_{j=1}^{r}[\hat{F}_n(x_{j-1}) - 1]^2 \log(1 - U_{(j)}) + \sum_{j=1}^{r}[\hat{F}_n(x_j) - 1]^2 \log(1 - U_{(j)})\}. \end{aligned}$$

Combining like terms finally yields the computing formula

$$\begin{aligned} A^2_{r,n} &= -n + n \textstyle\sum_{j=1}^{r} \{[\hat{F}_n(x_{(j-1)})]^2 - [\hat{F}_n(x_{(j)})]^2\} \log(U_{(j)}) \\ &\quad - \{[\hat{F}_n(x_{(j-1)}) - 1]^2 - [\hat{F}_n(x_{(j)}) - 1]^2\} \log(1 - U_{(j)}). \end{aligned} \tag{6}$$

Since the Kaplan-Meier estimator reduces to the EDF, both of the modified distance measures given above reduce to their complete sample counterparts in the case of no censoring.

2.1.3 Estimating the Parameters of the 3-Parameter Weibull distribution. The new minimum distance estimation techniques for randomly censored samples developed in Section 2.1.2 is applied in this section to the estimation of the parameters of the 3-parameter Weibull distribution. Using a tandem of minimum distance estimation and maximum likelihood estimation has been shown to be quite effective and robust in the estimation of the parameters of the 3-parameter Weibull distribution for complete samples [52, 64]. The success of a 1990 study by Gallagher and Moore [52] using minimum distance methods to estimate the location parameter and then maximum likelihood to estimate the shape and scale given the location parameter prompted this study for a

randomly censored Weibull. The parameterization of the Weibull distribution used here is

$$F(x) = 1 - e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}}, \quad x > \gamma, \eta > 0, \beta > 0$$

where γ, η and β represent the location, scale, and shape parameters, respectively. In the current study, our estimation procedure is as follows:

1. obtain an initial estimate of the location parameter using either $\hat{\gamma}^* = 0.999x_{(1)}$ or

$$\hat{\gamma}^* = x_{(1)} - \frac{\sum_{i=2}^{n}(x_i - x_{(1)})}{n(r-1)};$$

2. use ML to obtain initial estimates of shape and scale parameters;
3. use MDE with either $W_{r,n}^2$ or $A_{r,n}^2$ to refine the location parameter estimate given the shape and scale parameters;
4. re-estimate the shape and scale using ML estimation given the MD-refined location estimate.

The idea of using an initial location parameter estimate $\hat{\gamma}^* = 0.999x_{(1)}$ is from [5]. The factor of 0.999 is needed simply to avoid division by zero in the Anderson-Darling statistic. The idea of using

$$\hat{\gamma}^* = x_{(1)} - \frac{\sum_{i=2}^{n}(x_i - x_{(1)})}{n(r-1)}$$

was suggested by committee member Albert H. Moore and was inspired by Kapur and Lamberson [71] using it to estimate the location parameter of the 2-parameter exponential distribution.

The probability density function of the 3-parameter Weibull distribution is

$$f(x;\gamma,\beta,\eta) = \frac{\beta}{\eta^{\beta}}(x-\gamma)^{\beta-1}e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}}.$$

The likelihood function for an observed random sample $x_i = \min\{t_i, c_i\}, i = 1, \ldots, n$, with the indicator $\delta_i = I_{[t_i \leq c_i]}$ is given by

$$L(x;\gamma,\beta,\eta) = \prod_{i=1}^{n} \left\{ \frac{\beta}{\eta^{\beta}}(x_i-\gamma)^{\beta-1}\exp\left[-\left(\frac{x-\gamma}{\eta}\right)^{\beta}\right]\right\}^{\delta_i} \left\{\exp\left[-\left(\frac{x-\gamma}{\eta}\right)^{\beta}\right]\right\}^{1-\delta_i}.$$

The natural logarithm of the likelihood function is

$$\log L(x;\gamma,\beta,\eta) = \sum_{i=1}^{n} \delta_i \log\beta - \sum_{i=1}^{n} \delta_i \beta \log\eta + (\beta-1)\sum_{i=1}^{n} \delta_i \log(x_i-\gamma) - \frac{1}{\eta^{\beta}}\sum_{i=1}^{n}(x_i-\gamma)^{\beta}$$

and, letting $r = \sum_{i=1}^{n} \delta_i$, the partial derivatives with respect to the parameters are

$$\frac{\partial \log L(x;\gamma,\beta,\eta)}{\partial\gamma} = -(\beta-1)\sum_{i=1}^{n}\frac{\delta_i}{x_i-\gamma} + \frac{r\beta\sum_{i=1}^{n}(x_i-\gamma)^{\beta-1}}{\sum_{i=1}^{n}(x_i-\gamma)^{\beta}}, \tag{7}$$

$$\frac{\partial \log L(x;\gamma,\beta,\eta)}{\partial\eta} = -\frac{r\beta}{\eta} + \frac{\beta}{\eta^{\beta-1}}\sum_{i=1}^{n}(x_i-\gamma)^{\beta} \tag{8}$$

and

$$\frac{\partial \log L(x;\gamma,\beta,\eta)}{\partial\beta} = \frac{r}{\beta} - r\log\eta + \sum_{i=1}^{n}\delta_i\log(x_i-\gamma) - \sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\eta}\right)^{\beta}\log\left(\frac{x_i-\gamma}{\eta}\right). \tag{9}$$

The necessary and sufficient conditions for maximizing the likelihood function, or its natural logarithm, are well known. In order to find maximum likelihood estimates $\hat{\gamma}$, $\hat{\eta}$, and $\hat{\beta}$, it is necessary that Equations 7, 8, and 9 be set to zero and solved simultaneously. Tha sufficient condition is that the second partial derivatives are zero, which is easily verified. Upon setting each of the above equations equal to zero, it is possible to express $\hat{\eta}$ in terms of $\hat{\gamma}$ and $\hat{\beta}$ as

$$\hat{\eta} = \left[\frac{\sum_{i=1}^{n}(x_i-\hat{\gamma})^{\hat{\beta}}}{r}\right]^{\frac{1}{\hat{\beta}}} \tag{10}$$

but there are no explicit solutions for $\hat{\gamma}$ or $\hat{\beta}$. In fact, the maximum likelihood approach to finding a parameter estimate for γ often fails to converge, which is why we are using the highly reliable minimum distance method to estimate γ. Nevertheless, when the underlying distribution is correctly specified, maximum likelihood estimators are unbiased and minimum variance estimators and so we continue to estimate η and β using the likelihood equations. Now, the expression for $\hat{\eta}$ in Equation 10 can be substituted into Equation 9 to yield

$$\frac{r}{\hat{\beta}} + \sum_{i=1}^{n} \delta_i(x_i - \hat{\gamma}) - \frac{r \sum_{i=1}^{n} (x_i - \hat{\gamma})^{\hat{\beta}} \log(x_i - \hat{\gamma})}{\sum_{i=1}^{n} \delta_i (x_i - \hat{\gamma})^{\hat{\beta}}} = 0. \tag{11}$$

Once an initial estimate of the location parameter γ is obtained, Equations 10 and 11 are solved numerically for $\hat{\eta}$ and $\hat{\beta}$. The Newton-Raphson procedure was used for this purpose throughout this dissertation.

A Monte Carlo study was conducted to compare the four variations of this MD/ML estimation technique for the 3-parameter Weibull distribution. The censoring distribution in each case was a 2-parameter exponential distribution with its location parameter equal to that of the underlying Weibull distribution and scale parameter adjusted to give the desired expected proportion of censoring for each case. The estimation procedure is location and scale invariant, as discussed in Section 3.4.2, so the comparison was made for shape parameters 2 and 3.5. See Table 1 for the parameter values used in this study. The variations are distinguished by the choice of distance estimator and initial location parameter estimate and are abbreviated as shown in Table 2. Tabulated in Tables 3 and 4 are the mean and standard deviation (shown in parentheses) of each parameter as well as the integrated squared error (ISE) of the distribution function estimate. ISE is traditionally defined as the integrated squared difference between the estimated and true probability density functions, but the discrepancy between the estimated and true distribution functions will be used here instead. There are two reasons for making this change. First, some of the estimators used in the case of randomly censored data, such as the Kaplan-Meier estimator and the mean order

Table 1 Parameter Values of the 3-parameter Weibull for MISE Comparison.

Failure Distribution	Parameter Values	Censoring Distribution	Parameter Values	Expected Censoring
Weibull	$\beta = 2, \eta = 50, \gamma = 20$	Exponential	$\theta = 148, \gamma = 20$	$q = .25$
			$\theta = 57.75, \gamma = 20$	$q = .50$
			$\theta = 25.7, \gamma = 20$	$q = .75$
Weibull	$\beta = 3.5, \eta = 50, \gamma = 20$	Exponential	$\theta = 154, \gamma = 20$	$q = .25$
			$\theta = 62.5, \gamma = 20$	$q = .50$
			$\theta = 30, \gamma = 20$	$q = .75$

number estimator, are step functions of the distribution functions and provide no estimate of the density function. Second, the Cramér-von Mises and Anderson-Darling goodness-of-fit statistics use the distribution function rather than the density function to test the fit of a hypothesized parametric family.

The ISE of a distribution function estimator is defined here as

$$ISE(\hat{F}) = \int_{-\infty}^{+\infty} [\hat{F}(x) - F(x)]^2 dx$$

and was chosen because it provides a good overall measure of the closeness of an estimator to the true function. In the comparison procedure, the actual integration limits used were $\max\{\gamma, \hat{\gamma}\}$ for the lower limit and $x_{(n)} + 200$ for the upper limit. Monte Carlo samples of size $N = 1000$ were used to compare estimation techniques for samples of size $n = 20$ and 60 with expected proportion of censoring set at $q = 0.25, 0.50$, and 0.75. The ISE was computed for each of the 1000 Monte Carlo samples and the mean and standard deviation was found. The mean and standard deviation of each parameter estimate were also determined and are shown in Tables 3 and 4 along with the mean and standard deviation of the ISE.

The resulting mean and standard deviation (shown in parentheses) shown in Tables 3 and 4 for each parameter estimate and the ISE indicate that although the modified Anderson-Darling statistic performs slightly better than the modified Cramér-von Mises statistic in the majority of cases according to MISE criterion, the estimators are very similar. Furthermore, the choice of the

Table 2 Key to Abbreviations of MD/ML Estimators for the 3-Parameter Weibull Distribution.

Abbreviation	Description
MDLCvM1	Minimum Distance for the Location using the Cramér-von Mises statistic with initial estimate $\hat{\gamma}^* = 0.999x_{(1)}$ and ML estimation for shape and scale parameters
MDLAD1	Minimum Distance for the Location using the Anderson-Darling statistic with initial estimate $\hat{\gamma}^* = 0.999x_{(1)}$ and ML estimation for shape and scale parameters
MDLCvM2	Minimum Distance for the Location using the Cramér-von Mises statistic with initial estimate $\hat{\gamma}^* = x_{(1)} - \frac{\sum_{i=2}^{n}(x_i - x_{(1)})}{n(r-1))}$ and ML estimation for shape and scale parameters
MDLAD2	Minimum Distance for the Location using the Anderson-Darling statistic with initial estimate $\hat{\gamma}^* = x_{(1)} - \frac{\sum_{i=2}^{n}(x_i - x_{(1)})}{n(r-1))}$ and ML estimation for shape and scale parameters

initial location parameter estimate does not seem to have much of an effect, demonstrating the robustness of MDE even when samples are randomly censored. As expected, however, parameter and distribution function estimation improves with increased sample size but is increasingly hindered as censoring increases. Another important result found in Tables 3 and 4 is the effect of censoring on the parameter estimates. The trend in the parameter estimation seen here is that location parameter estimates become increasingly lower while shape and scale parameter estimates creep ever higher as the expected proportion of censoring increases. A repetition of steps 3 and 4 of our estimation procedure on page 18 of this document was examined for the 3-parameter Weibull distribution with shape values 2 and 3.5 and was found to provide no additional improvement in estimation.

Because ML estimates are asymptotically normally distributed, the mean and standard deviation are suitable comparison statistics. In contrast, the inherent skewness of ISE, see Figure 1 for example, makes it difficult to compare these estimators based on the mean and standard deviation of the ISE alone. A more suitable comparison may be obtained using the Kruskal-Wallis test for homogeneity, a nonparametric procedure used to determine if values of one population are systematically larger than the values of another [85: pp. 409-410]. Several populations may be compared simultaneously. Kruskal-Wallis test results and side-by-side boxplots can be used to

Table 3 3-Parameter Weibull, shape $\beta = 2$, with Exponential Censoring.
Mean and Standard Deviation (in parantheses)

Expected Censoring	Sample Size	Method of Estimation	Location $\gamma = 20$	Scale $\eta = 50$	Shape $\beta = 2$	MISE of CDF (std. dev.)
$q = 0.25$	$n = 20$	MDLCvM1	24.50 (4.12)	44.29 (7.89)	1.76 (0.57)	0.6467 (0.6401)
		MDLAD1	24.42 (3.96)	44.52 (7.60)	1.85 (0.50)	0.5556 (0.5904)
		MDLCvM2	22.83 (4.58)	46.40 (8.28)	1.94 (0.55)	0.5561 (0.5890)
		MDLAD2	22.27 (2.24)	47.17 (7.97)	2.04 (0.50)	0.5284 (0.5846)
	$n = 60$	MDLCvM1	21.78 (1.84)	47.85 (4.39)	1.91 (0.28)	0.1882 (0.2074)
		MDLAD1	21.85 (1.75)	47.81 (4.29)	1.92 (0.26)	0.1813 (0.2036)
		MDLCvM2	21.29 (2.02)	48.45 (4.53)	1.94 (0.28)	0.1845 (0.2063)
		MDLAD2	21.15 (1.82)	48.66 (4.36)	1.97 (0.25)	0.1765 (0.2027)
$q = 0.50$	$n = 20$	MDLCvM1	21.48 (3.47)	48.35 (9.64)	2.13 (0.71)	0.9028 (1.0323)
		MDLAD1	21.58 (3.26)	48.09 (9.49)	2.15 (0.67)	0.8429 (1.0085)
		MDLCvM2	19.03 (4.12)	50.82 (10.29)	2.35 (0.73)	0.8351 (1.0139)
		MDLAD2	18.80 (3.71)	51.02 (10.06)	2.39 (0.70)	0.8216 (1.0097)
	$n = 60$	MDLCvM1	20.49 (1.09)	49.48 (4.90)	2.03 (0.31)	0.2671 (0.3194)
		MDLAD1	20.60 (0.96)	49.34 (4.85)	2.03 (0.30)	0.2629 (0.3172)
		MDLCvM2	18.77 (1.35)	50.27 (5.07)	2.08 (0.32)	0.2640 (0.3179)
		MDLAD2	19.75 (1.09)	50.27 (4.97)	2.09 (0.30)	0.2597 (0.3163)
$q = 0.75$	$n = 20$	MDLCvM1	16.04 (7.11)	53.56 (17.46)	2.95 (2.12)	1.8616 (6.1219)
		MDLAD1	16.30 (6.98)	53.28 (17.39)	2.94 (2.16)	1.8534 (6.1202)
		MDLCvM2	12.58 (7.15)	57.00 (18.09)	3.34 (2.51)	1.9296 (6.6844)
		MDLAD2	12.53 (6.91)	56.95 (17.65)	3.36 (2.51)	1.9127 (6.6194)
	$n = 60$	MDLCvM1	18.25 (3.39)	51.80 (9.08)	2.28 (0.47)	0.5704 (0.7968)
		MDLAD1	18.44 (3.15)	51.61 (8.93)	2.27 (0.46)	0.5719 (0.8013)
		MDLCvM2	17.2 (3.64)	52.78 (9.24)	2.37 (0.50)	0.5730 (0.7912)
		MDLAD2	17.33 (3.40)	52.63 (9.10)	2.37 (0.49)	0.5716 (0.7900)

Table 4 3-Parameter Weibull, shape $\beta = 3.5$, with Exponential Censoring.
Mean and Standard Deviation (in parantheses)

Expected Censoring	Sample Size	Method of Estimation	Location $\gamma = 20$	Scale $\eta = 50$	Shape $\beta = 3.5$	MISE of CDF (std. dev.)
$q = 0.25$	$n = 20$	MDLCvM1	26.41 (5.96)	42.83 (7.38)	3.17 (1.06)	0.4016 (0.4242)
		MDLAD1	26.47 (5.84)	42.79 (7.19)	3.20 (0.99)	0.3671 (0.3931)
		MDLCvM2	24.70 (6.55)	44.66 (7.97)	3.36 (1.09)	0.3791 (0.4048)
		MDLAD2	24.45 6.27)	44.99 (7.60)	3.42 (1.02)	0.3514 (0.3851)
	$n = 60$	MDLCvM1	22.25 (2.40)	47.58 (3.37)	3.40 (0.47)	0.1122 (0.1292)
		MDLAD1	22.34 (2.36)	47.49 (3.33)	3.39 (0.47)	0.1116 (0.1285)
		MDLCvM2	21.71 (2.54)	48.15 (3.51)	3.45 (0.48)	0.1120 (0.1291)
		MDLAD2	21.70 (2.46)	48.17 (3.43)	3.45 (0.47)	0.1107 (0.1280)
$q = 0.50$	$n = 20$	MDLCvM1	22.55 (3.33)	46.98 (5.80)	3.82 (1.26)	0.5352 (0.5813)
		MDLAD1	22.71 (3.24)	46.80 (5.73)	3.81 (1.25)	0.5260 (0.5710)
		MDLCvM2	19.88 (3.98)	49.74 (6.55)	4.11 (1.38)	0.5252 (0.5738)
		MDLAD2	19.81 (3.74)	49.81 (6.37)	4.12 (1.35)	0.5197 (0.5709)
	$n = 60$	MDLCvM1	20.71 (1.15)	49.15 (2.90)	3.58 (0.53)	0.1564 (0.1875)
		MDLAD1	20.84 (1.08)	49.03 (2.86)	3.57 (0.52)	0.1563 (0.1875)
		MDLCvM2	19.95 (1.34)	49.95 (3.04)	3.66 (0.54)	0.1562 (0.1876)
		MDLAD2	19.99 (1.20)	49.91 (2.97)	3.65 (0.54)	0.1562 (0.1875)
$q = 0.75$	$n = 20$	MDLCvM1	19.54 (4.12)	49.83 (8.60)	4.75 (2.95)	0.9923 (1.1529)
		MDLAD1	19.68 (4.09)	49.67 (8.53)	4.74 (2.98)	0.9915 (1.1526)
		MDLCvM2	14.25 (5.60)	55.11 (9.91)	5.50 (3.51)	0.9943 (1.1598)
		MDLAD2	14.26 (5.48)	55.09 (9.87)	5.50 (3.51)	0.9938 (1.1603)
	$n = 60$	MDLCvM1	19.85 (1.33)	50.02 (4.46)	3.78 (0.78)	0.3329 (0.4298)
		MDLAD1	19.97 (1.25)	49.88 (4.43)	3.77 (0.77)	0.3327 (0.4293)
		MDLCvM2	18.58 (1.75)	51.31 (4.78)	3.92 (0.82)	0.3326 (0.4303)
		MDLAD2	18.63 (1.59)	51.25 (4.71)	3.92 (0.81)	0.3322 (0.4296)

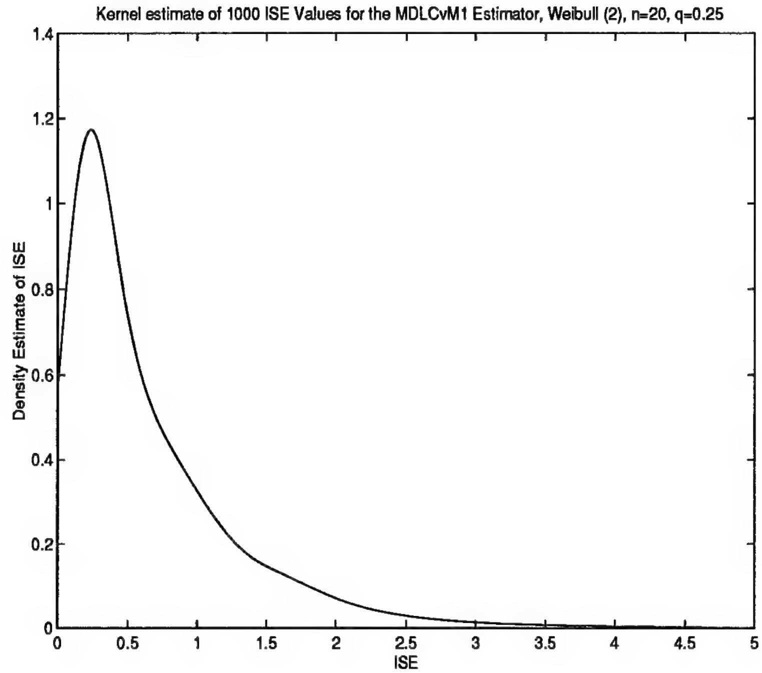

Figure 1 An Example of the Skewness of Integrated Squared Error at Sample Size 20.

statistically discern which estimator has the lowest ISE under a given set of conditions (i.e. amount of censoring, sample size, and underlying distribution).

For each Weibull distribution, level of censoring and sample size, Kruskal-Wallis tests were used to compare Monte Carlo samples of 1000 ISE measurements for each of the four MD/ML estimation techniques. P-values of the Kruskal-Wallis tests are given in Table 5. Because a higher p-value indicates stronger evidence supporting homogeneous populations, there is little doubt that the four MD/ML estimation techniques provide statistically equivalent estimates under each circumstance, with the exception of the Weibull with shape 2 at sample size 20 with 25% censoring. With a p-value of only 0.14, which is displayed in boldface type in Table 5, further investigation of the estimators was warranted. Pairwise comparisons of the four MD/ML estimation techniques at $n = 20$ and $q = 0.25$ for the Weibull with shape 2 revealed one significant difference: the MDLAD2 technique had significantly lower ISE than the MDLCvM1 estimation technique (p-value $\approx$ 0). This does not, however, imply that the ISE of MDLAD2 is significantly lower than the ISE of

Table 5 P-values of the Kruskal-Wallis Test for ISE Comparison of MD/ML Estimators. 3-Parameter Weibull with Exponential Censoring

	n	For $\beta = 2$	For $\beta = 3.5$
$q = 0.25$	20	**0.14**	0.70
	60	0.85	1.00
$q = 0.50$	20	0.59	1.00
	60	0.95	1.00
$q = 0.75$	20	0.96	1.00
	60	1.00	1.00

MDLCvM2 or MDLAD1, nor does it imply that the ISE of MDLCvM1 is significantly higher than the ISE of MDLCvM2 or MDLAD1, as no other significant differences were found among any other pairwise comparisons.

2.2 Nonparametric Estimation

2.2.1 The Kaplan-Meier Product-Limit Estimator. One of the earliest works to recognize the problem of statistical analysis in the presence of random censorship, particularly as it relates to the theory of medical follow-up studies, is by Harris, Meier and Tukey [60]. However, actuarial life table estimation of a distribution function from variably censored observations was performed as early as 1912 using arbitrary grouping intervals [70]. In 1926 Greenwood derived the variance of the life table estimate, which was extended in 1949 by Irwin to obtain an estimated covariance function [70]. Kaplan and Meier [70] showed in their famous 1958 paper that the arbitrary groupings were unnecessary, thus introducing the product-limit estimator, which is probably the most well-known and possibly the most significant contribution to the field regarding randomly censored data. The product-limit estimator is thought of as the nonparametric maximum likelihood estimator of the underlying survivor function and reduces to the empirical distribution function (EDF) when no censoring occurs. The terms "product-limit estimator" and "Kaplan-Meier estimator" are used interchangeably. Efron showed the product-limit estimator to be "self-consistent", among other things, in this often referenced contribution to the 1967 Berkeley Symposium on Mathematical Statistics and Probability [43].

Many authors have examined the properties of the Kaplan-Meier estimator (KME). Breslow and Crowley [13] established limiting pointwise normality for the product-limit estimate. Confidence bands for the survival probabilities given by the KME have been explored by Thomas and Grunkemeier [134], Gillespie and Fisher [55] and Hall and Wellner [59]. Meier [97] provides proofs of many properties of the KME, including the Markovian property, Greenwood's asymptotic variance formulation, consistency and asymptotic normality, and illustrates the parallel to properties of the EDF for complete samples. Peterson [7] demonstrates the utility of expressing the KME as a function of empirical subsurvival functions by using them to give an easy proof of the strong consistency of the KME. Eubank and LaRiccia [45] used a regression model for the Kaplan-Meier quantile process to construct asymptotically normal and efficient estimators of location and scale parameters. Chen, Hollander and Langberg [23] address small sample results for the KME under the proportional hazards model used by Koziol and Green [81]. Miller [99] looked at the asymptotic efficiency of the KME relative to the maximum likelihood estimator (MLE) of the survival function. The KME and MLE of the survival function are compared for samples containing undetected outliers by Aranda-Ordaz [105].

Mauro [93] considers the KME from a combinatoric viewpoint and Wellner [142] provides an approximate variance formula for the KME which is compared to that given by Chen, Hollander and Langberg [23]. Wang [139] showed the uniform consistency of the KME and Kumazawa [82] derived a consistent estimator of the variance function, which is dependent on the censoring distribution, of a life expectancy estimator based on the KME given by Yang [152]. Zhou [153] examines the effects of non-i.i.d. survival and censoring times on the KME. Stute and Wang [124] prove the strong law under random censorship, which can be used to show consistency of many estimators. Stute [126] shows the bias of Kaplan-Meier integrals. Stute and Wang [126] provide a jackknife estimate of the Kaplan-Meier integral and use it to derive formulas for mean life, higher moments, and mean residual life. The Kaplan-Meier integral is proven to be asymptotically normal when properly standardized by Stute [125] using the Central Limit Theorem. Csorgo [30] establishes

universal Gaussian approximations for empirical cumulative hazard and product-limit processes under random censorship. Van Keilegom and Veraverbeke [73] investigated almost sure convergence properties of the KME.

In defining the Kaplan-Meier product-limit estimator, we consider the random sample $x_i = \min(t_i, c_i), i = 1, \ldots, n$, with the indicator $\delta_i = I_{[t_i \leq c_i]}$ as defined in Chapter I. The product-limit estimator of the distribution function is defined by

$$\hat{F}_n(x) = \begin{cases} 0, & x < x_{(1)} \\ 1 - \prod_{i:x_i \leq x} [\frac{n - R_i}{n - R_i + 1}]^{\delta_i}, & x < x_{(n)} \\ 1, & x > x_{(n)} \end{cases}$$

where R_i is the rank of the pair $(x_i, 1 - \delta_i)$ in the lexicographic ordering of the sequence $(x_1, 1 - \delta_1), (x_2, 1 - \delta_2), \ldots, (x_n, \delta_n)$. Notice that this formulation treats the last observation as a failure regardless of whether it actually is or not. This practice has been adopted by many authors since the estimator was first put forth by Kaplan and Meier [70] in 1958, who stated at the time that the product-limit estimator should not be used beyond $x_{(n)}$ if $x_{(n)}$ corresponded to a censoring time. They simply say that $\hat{F}_n(x)$ should be considered to fall between 0 and $\hat{F}_n(x_{(n)})$ [70: p. 463]. Plots of examples illustrating the Kaplan-Meier estimator can be found in Appendix C.

2.2.2 The Mean Order Number Estimator. The idea here is to estimate the true order number of each observed failure since the order number it is assigned in the randomly censored sample is usually not the order number it would have assumed had the censored items been observable until failure. To estimate the true order number consider all possible combinations of censored and failed items in which a particular failed item could have been assigned a certain order number, then use the number of ways each item could take on each particular order number to find the mean order number. The following algorithm is outlined in Kapur and Lamberson [71] and Sun and Kececioglu [127]. Let $O_i, i = 1, \ldots, r$, represent the mean order number for ordered failure

number i and N^+ denote the number of items following the current set of withdrawals. Now, if the first observation in the full randomly censored data set is an observed failure, then $O_1 = 1$, if the second observation is also an observed failure then $O_2 = 2$, and so on until the first withdrawal. Once censoring times have been encountered in the ordered set of failures and withdrawals, the formula [71] for computing the increment, I, needed to estimate the order number of the next failure time is

$$I_j = \tfrac{n+1-O_{j-1}}{1+N^+}, \quad j = 1, \ldots, r,$$

where $O_0 = 0$ if the first item in the set is censored. Note that if there are no withdrawals between the $(j-1)^{st}$ and j^{th} failures, then $I_j = I_{j-1}$. The estimated order number for the $(j+1)^{st}$ failure is

$$O_{j+1} = O_j + I_j.$$

From the estimated order numbers, an estimator of the distribution function can be defined as [127]

$$\hat{F}_{MONE}(t) = \begin{cases} 0, & t < t_{(0)} \\ \dfrac{O_{(j-1)}}{n}, & t_{(j-1)} \leq t < t_{(j)}, j = 1, \ldots, r+1 \\ 1, & t > t_{(r+1)} \end{cases}$$

where $O_0 = 0, t_{(0)} = 0$, and $t_{(r+1)}$ is a pseudo-failure, possibly defined to be $t_{(r)}, +\infty$, or somewhere in between. Note that other plotting positions based on the mean order numbers may be used to form the estimator. Examples of the mean order number estimator are in Appendix C.

2.2.3 The Piecewise Exponential Estimator. In 1983 Kitchen, Langberg, and Proschan [76] published an article introducing the Piecewise Exponential Estimator (PEXE). This continuous alternative to the Kaplan-Meier and mean order number estimators is shown to be strongly consistent under a mild regularity condition on the distribution of the censoring variable and its errors are said to follow the same Gaussian process as those associated with the Kaplan-Meier estimator [76].

Further discussions of the properties of the PEXE as well as a comparison to Kaplan-Meier estimator can be found in [74] and [143]. The development and rationale for the PEXE are as follows. A set of n items are placed on test at time $t = 0$ and observed until failure or withdrawal, whichever occurs first. Let t_i represent the ordered observed failure times, $c_{i,j}$ represent the ordered censoring times, and k_i be the number of censored observations between consecutive observed failures. Since both failure and censoring distributions are assumed to be continuous, no ties will be considered. Thus,

$$\begin{aligned} 0 &< c_{1,1} < \cdots < c_{1,k_1} < t_1 < c_{2,1} < \cdots c_{2,k_2} < t_2 < \cdots \\ &< t_{r-1} < c_{r,1} < \cdots c_{r,k_r} < t_r < c_{r+1,1} < \cdots < c_{r+1,k_{r+1}}. \end{aligned}$$

An estimate of the failure rate in the interval $(t_{i-1}, t_i]$, where $i = 1, 2, \ldots, r$, and $t_0 = 0$, is the number of failures observed in the interval, which is 1, divided by the total time on test in that interval. This procedure estimates a constant failure rate

$$\hat{z}_i = \frac{1}{\sum_{j=1}^{k_i}(c_{i,j} - t_{i-1}) + (n - i + 1 - \sum_{j=1}^{i} k_j) \cdot (t_i - t_{i-1})}, \quad i = 1, \ldots, r,$$

for each interval between successive failures which is then used to fit an exponential estimator of the survivor function on each interval. The separate exponential survivor functions are pieced together to form the PEXE of the survivor function up to the r^{th} failure. Formally then, for a set of r ordered failure times from a randomly censored sample the PEXE of a survivor function is defined as

$$\hat{S}_{PEXE}(x) = \begin{cases} \exp(-\hat{z}_1 x) & 0 \le x \le t_1 \\ \exp\{-[\hat{z}_1 t_1 + \hat{z}_2(x - t_1)]\} & t_1 < x \le t_2 \\ \exp\{-[\hat{z}_1 t_1 + \hat{z}_2(t_2 - t_1) + \cdots + \hat{z}_i(x - t_{i-1})]\} & t_{i-1} < x \le t_i, i = 3, \ldots, r \\ \exp\{-[\hat{z}_1 t_1 + \hat{z}_2(t_2 - t_1) + \cdots + \hat{z}_r(x - t_r)]\} & x > t_r. \end{cases}$$

Some presentations of the PEXE do not define an estimate beyond t_r [74] while Westberg and Klefsjo [143] suggest using a Weibull distribution with shape parameter 2 and scale parameter, η estimated by

$$\hat{\eta} = \frac{t_r}{\sqrt{-\log(\hat{S}_{PEXE}(t_r))}}$$

if there is any indication that the data are from a distribution with an increasing failure rate. A wise choice of estimator to extrapolate beyond t_r is given by Sun and Kececioglu [127] and is used in the definition of the PEXE given above. That is, from t_r to $+\infty$ use an exponential fit with hazard rate equal to that of the neighboring interval t_{r-1} to t_r. Examples of the PEXE are given in Appendix C.

2.2.4 Kernel Estimators. A big problem with the nonparametric distribution function estimators for randomly censored samples presented so far is their lack of ability to extrapolate beyond the last observed failure. The kernel estimators are better in this area, particularly the one proposed by Földes, Rejtő, and Winter [49].

Blum and Susarla [11] were the first to generalize the kernel-type density estimator introduced by Rosenblatt in 1956 to estimate a density function from randomly right-censored data. The estimator is defined as

$$\hat{f}_{BSE}(x) = (nh_n)^{-1} \cdot \frac{\sum_{j=1}^{n} k(\frac{x - X_{(j)}}{h_n})\delta_j}{H_n^*(x)}$$

where

$$H_n^*(x) = \prod_{i:X_i \leq x} \left[\frac{n - R_i + 1}{n - R_i + 2}\right]^{1-\delta_i}$$

is a modified product-limit estimator of the censoring distribution, $k(t)$ is a kernel function satisfying the requirements of a probability density function, and $\{h_n\}$ is a positive bandwidth sequence such that $\lim_{n\to\infty} h_n = 0$. We selected the data-dependent bandwidth $h = s^* n^{-\frac{1}{5}}$ where s^* is the standard deviation of the failure set. The standard normal kernel was used throughout this work.

Naturally, the distribution function estimate is given by

$$\hat{F}_{BSE}(x) = \int_{-\infty}^{x} \hat{f}_{BSE}(t)dt.$$

Földes, Rejtő, and Winter [48,49] establish strong consistency of kernel-type estimators using convergence properties of the Kaplan-Meier product limit estimator. They defined a kernel-type density estimator that was very similar to the one proposed by Blum and Susarla. In fact, they are asymptotically equivalent [108]. However, the Földes, Rejtő, and Winter (FRW) estimator has better small sample properties and reduces to the usual Parzen-type kernel density estimator in the case of no censoring while the Blum and Susarla estimator does not. The FRW kernel density estimator is defined as

$$\hat{f}_{FRWE}(x) = \frac{1}{h_n}\sum_{i=1}^{n} \Delta_i k\left(\frac{x - X_{(i)}}{h_n}\right)$$

where Δ_i is the jump of the Kaplan-Meier estimator from X_{i-1} to X_i and h_n is defined as it is in the Blum-Susarla estimator. Illustrations of this estimator are given in Appendix C. The distribution function estimate is

$$\hat{F}_{FRWE}(x) = \int_{-\infty}^{x} \hat{f}_{FRWE}(t)dt.$$

Kernel estimation of the hazard rate function, which is closely related to the density function, is considered by Tanner and Wong [133]. Tanner also studied the effects of using a variable kernel estimator, in which the bandwidth is allowed to vary, and presented sufficient conditions for its strong consistency [132]. Yandell [151] derived simultaneous confidence bands for kernel-type estimators and extended global deviation and mean square deviation results to the case of randomly censored survival data. Burke and Horváth [17] prove various properties, such as strong consistency, of density and failure rate estimators based on the Kaplan-Meier estimator in a competing risks model and also provide limit theorems for some quadratic functionals of each. Padgett and

McNichols [109] provide a summary of density estimators for randomly censored data in which they outline all of the estimators developed by 1984.

Additionally, McNichols and Padgett [94] introduced an adaptive kernel density estimator for randomly right-censored data and discussed the convergence in probability and the almost sure convergence of the modified estimator. An asymptotic look at the decomposition of integrated squared error into variance and squared bias components is given by Marron and Padgett [92] in addition to a data-based method of selecting an asymptotically optimal bandwidth for kernel density estimators under random censorship. Padgett [108] presents and summarizes these nonparametric density estimators as well as histogram estimators for randomly censored data.

Although kernel density estimation is computationally intensive and somewhat cumbersome, it is an excellent way to gain insight into the general shapes of the density, distribution, survivor, and hazard functions of the true underlying process outside of the confines of a parametric model. It is also a very good way to get an impression of skewness and modality. One of the primary advantages that kernel estimators have over all of the other estimators is their ability to provide smooth, continuous estimates of density functions. These insights can be a valuable starting point for choosing an appropriate parametric distribution for goodness-of-fit testing.

2.2.5 New Trigonometrically-Smoothed and Jackknifed Estimators for Randomly Censored Data. In 1982 James Sweeder developed a class of trigonometrically-smoothed and jackknifed estimators of distribution and density functions based on the EDF under the supervision of Dr. Albert H. Moore in a doctoral dissertation for the Air Force Institute of Technology [131]. This class of estimators not only provides continuous estimates of distribution functions but can be differentiated to give estimates of density functions as well. Furthermore, Sweeder's estimators were shown to outperform the EDF according to MISE criterion, provide empirical convergence rates that are quite close to those of other methods of estimation for complete samples, and work well for relatively small sample sizes. Sweeder's estimator can be generalized to estimate both

distribution and density functions for randomly censored data by substituting the Kaplan-Meier estimator or the mean order number estimator in place of the EDF.

Analogous to Sweeder's estimator, we begin with a trigonometrically smoothed estimate of the distribution function making use of the cosine function as follows

$$\tilde{F}_n(x) = \begin{cases} 0, & x < X_{(0)} \\ G_i + \dfrac{G_{i+1} - G_i}{2}\left[1 - \cos\pi\left(\dfrac{x - X_{(i)}}{X_{(i+1)} - X_{(i)}}\right)\right], & X_{(i)} \le x < X_{(i+1)}, i = 0, \ldots, n \\ 1, & x > X_{(n+1)} \end{cases}$$

where $G_i, i = 1, \ldots, n$ represents a plotting position for the i^{th} ordered observation obtained via the Kaplan-Meier or the mean order number estimator and extrapolation points x_0 and x_{r+1} defined such that $G_0 = 0$ and $G_{n+1} = 1$. Differentiating leads to the density function estimator

$$\tilde{f}_n(x) = \begin{cases} \dfrac{\pi}{2}\dfrac{G_{i+1} - G_i}{X_{(i+1)} - X_{(i)}}\left[\sin\pi\left(\dfrac{x - X_{(i)}}{X_{(i+1)} - X_{(i)}}\right)\right], & X_{(i)} \le x < X_{(i+1)}, i = 0, \ldots, n \\ 0, & \text{elsewhere.} \end{cases} \tag{12}$$

The trigonometric smoothing creates a continuous estimate of the distribution function, but the density estimate still has problems because the derivative of the distribution function is zero at the observed failure points. This is where the jackknife comes in. Sweeder [131] uses a method proposed for bias reduction by Quenouille [113] to sidestep the zero derivative points. This variation of the jackknife approach involves the division of the sample distribution function into subsamples. This procedure is outlined by Sweeder [131: pp. 19-20] for complete samples.

For randomly censored samples, obtaining the subsamples can be achieved as follows. Let $t_{(1)}, \ldots, t_{(r)}$ be the set of ordered failure times from a randomly censored sample. The number of subsamples chosen is $k \le \frac{r}{2}$. Let $\ell = 1, \ldots, k$ index the subsamples and let $Y_{(j,\ell)}$ be the j^{th} element of subsample ℓ. Now, with k subsamples, $r = km + R$ where $m = \lfloor \frac{r}{k} \rfloor$ and $R = n \bmod k$. The

assignments are made to the subsamples as

$$Y_{(j,\ell)} = X_{(\ell+k(j-1))}$$

where

$$\begin{aligned} j &= 1, \dots, m && \text{if } \ell > R \\ j &= 1, \dots, m+1 && \text{if } \ell \leq R. \end{aligned}$$

This creates k ordered subsamples, R of size $m+1$ and $k-R$ of size m. Let

$$n^* = \begin{cases} m & \text{if } \ell > R \\ m+1 & \text{if } \ell \leq R. \end{cases}$$

In practice, extrapolation points for each subsample can be obtained by assigning

$$Y_{(0,\ell)} = \max\{2Y_{(1,\ell)} - Y_{(2,\ell)}, t_{(1)}\}$$

and

$$Y_{(n^*+1,\ell)} = \max\{2Y_{(n^*,\ell)} - Y_{(n^*-1,\ell)}, t_{(r)}\}.$$

Using each of the k subsamples, we construct continuous, differentiable estimates of the distribution function through equation 5. Finally, the sample distribution function estimator is

$$F_n^*(x) = \begin{cases} 0, & x < X_{(1)} \\ \frac{1}{k}\sum_{\ell=1}^{k} \tilde{F}_{\ell,n^*}(x), & X_{(1)} \leq x < X_{(n)} \\ 1, & x > X_{(n)}. \end{cases} \tag{13}$$

Similarly, the density estimator is defined as

$$f_n^*(x) = \begin{cases} \frac{1}{k}\sum_{\ell=1}^{k} \tilde{F}'_{\ell,n^*}(x), & X_{(1)} \leq x < X_{(n)} \\ 0, & \text{elsewhere.} \end{cases}$$

Plots illustrating the use of this estimator with and without the jackknifing are given in Appendix C.

2.3 Semi-Parametric Estimation

2.3.1 The Klein, Lee, and Moeschberger Partially Parametric Estimator. In 1990 Klein, Lee and Moeschberger [78] introduced an estimator of the survivor function (and hence the distribution function) for randomly censored data. Here it will be refered to as the KLM estimator. This estimator is relatively easy to construct and compares favorably to the completely nonparametric estimators. For the parametric part of this partially parametric estimator, we let $R(x \mid \theta)$ denote a parametric survivor function where θ is a vector of unknown parameters estimated by some consistent estimator $\hat{\theta}$ based on the observed data. One could inspect a kernel density estimate or a product-limit estimate of the function or perhaps use some prior knowledge of a process to select a reasonable parametric model.

Recall that in the competing risks model of random censorship $x_i = \min\{t_i, c_i\}$ and $\delta_i = I_{t_i \leq c_i}$. For some arbitrary value x, there are four distinct possibilities: (1) the observed value x_i is greater than x and represents an observed failure, (2) the observed value of x_i is greater than x and corresponds to a censoring time, (3) x_i is less than x and represents a failure, and (4) x_i is less than x and represents a censored observation. The scenarios represented by cases (1), (2), and (3) can all be handled nonparametrically. However, case (4) presents the need for some way to estimate the probability that an item whose lifetime will ultimately be censored would have survived beyond x. The parametric model is used for this. With that in mind, the KLM partially parametric estimator

of a survivor function of a randomly censored sample of size n is given by

$$\hat{S}_{KLM}(x) = \frac{\sum_{i=1}^{n} \phi_i(x \mid R(\cdot \mid \hat{\theta}))}{n}$$

where

$$\phi_i(x \mid R(\cdot \mid \hat{\theta})) = \begin{cases} 1 & \text{if } X_i > x \text{ and } \delta_i = 0 \text{ or } 1 \\ 0 & \text{if } X_i \leq x \text{ and } \delta_i = 1 \\ \frac{R(x \mid \hat{\theta})}{R(X_i} \mid \hat{\theta}) & \text{if } X_i \leq x \text{ and } \delta_i = 0. \end{cases}$$

This estimator reduces to the usual complete sample EDF in the case of no censoring. Further, it can be seen that the greater the proportion of censoring, the more the estimator will resemble the parametric estimator $R(x \mid \hat{\theta})$. Examples can be found in Appendix C. In fact, when $R(x \mid \theta)$ is chosen correctly, the estimator converges uniformly if the parameter estimates converge, such as when maximum likelihood is used [78]. Klein, Lee, and Moeschberger [78] also state that the process $\sqrt{n}(\hat{S}_{KLM}(x) - S(x))$ converges weakly to a zero mean Gaussion process. However, the covariance of this process is very difficult to determine for many distributions. They [78] were able to derive the covariance function in the case of an exponential failure distribution subject to an exponential censoring distribution and found that the asymptotic variance of the KLM estimator is always smaller than that of the Kaplan-Meier estimator, assuming the parametric part of the model is chosen correctly.

2.3.2 A Semi-Parametric Kaplan-Meier Estimator. Since the Kaplan-Meier estimator provides no estimate of a survivor function beyond the maximum observation when it corresponds to a censoring time, some authors [77,117] use a parametric distribution for this purpose. Although any appropriate distribution may be chosen for a given situation, the Weibull distribution is generally accepted as providing a reasonable fit to lifetime data. For the Kaplan-Meier estimator, if the maximum observation is an observed failure time, then $\hat{S}_n = 0$ for $x > x_{(n)}$. In 1998, Rufflin [117]

used the two-parameter Weibull distribution to estimate $S(x)$ for $x > x_{(n)}$ when $x_{(n)}$ represents a censoring time, using maximum likelihood to provide parameter estimates. Klein and Moeschberger [77] used the same approach but found that tying the estimators together at $x_{(n)}$ by maximizing the likelihood function with respect to the parameters of the two-parameter Weibull distribution subject to the constraint

$$e^{-\left(\frac{x_{(n)}}{\eta}\right)^{\beta}} = \hat{S}_n(x_{(n)})$$

where η and β are the scale and shape, respectively. This practice will also guarantee a monotonic estimate, whereas the "untied" version may not. Examples are given in Appendix C.

2.4 A Comparison of Distribution Function Estimators

A comparison of estimation techniques was conducted using the mean integrated squared error (MISE) as well as nonparametric Kruskal-Wallis tests on integrated squared error (ISE) of the estimated cumulative distribution function (CDF) for each. MISE is a measure of discrepancy between the true CDF and the estimated CDF and so an estimator yielding a smaller value is a better estimator under the given circumstances. Furthermore, the standard deviation (shown in parentheses) of the integrated squared error was obtained to give some idea of the precision of the Monte Carlo simulation. 1000 samples of sizes 20 and 60 were evaluated at 0.25, 0.50, and 0.75 expected proportion of censoring. Numerical integration was performed using Simpson's rule with 1001 function evaluations. For the exponential and 2-parameter Weibull distributions, a lower limit of 0 and an upper limit of $x_{(n)} + 200$ was used. For the 3-parameter Weibull the lower limit of integration was chosen to be the maximum of the true location parameter value and its estimate. The upper limit was $x_{(n)}$ for each sample. Since distribution function estimation beyond $x_{(n)}$ presents difficulties for some estimators, $x_{(n)} + 200$ seemed an appropriate choice for the upper integration limit with the parameter values used here to compare each estimator's ability to extrapolate beyond $x_{(n)}$ in the case of randomly censored data.

The study was conducted for the exponential distribution, two cases of the 2-parameter Weibull (location known) with shape parameter values 2 and 3.5, and two cases of the 3-parameter Weibull also with shape parameter values 2 and 3.5. The censoring distribution in all cases was chosen to be an exponential distribution with its parameters adjusted to provide the desired expected proportion of censoring. Table 4 shows the parameter combinations used for each case. All of

Table 6 Parameter Values for MISE Comparison.

Failure Distribution	Parameter Values	Censoring Distribution	Parameter Values	Expected Censoring
Exponential	$\eta = 50$	Exponential	$\theta = 150$ $\theta = 50$ $\theta = 16\frac{2}{3}$	$q = .25$ $q = .50$ $q = .75$
Weibull	$\beta = 2, \eta = 50$	Exponential	$\theta = 148$ $\theta = 57.75$ $\theta = 25.7$	$q = .25$ $q = .50$ $q = .75$
Weibull	$\beta = 3.5, \eta = 50$	Exponential	$\theta = 154$ $\theta = 62.5$ $\theta = 30$	$q = .25$ $q = .50$ $q = .75$

Table 7 Key to Abbreviations of Estimators.

Abbreviation	Description
MLE	Maximum Likelihood Estimation
KME	Kaplan-Meier Estimation
TS-KME	Trigonometrically-Smoothed KME
TSJ-KME	Trigonometrically-Smoothed and Jackknifed KME
MONE	Mean Order Number Estimation
PEXE	Piecewise Exponential Estimation
FRWE	Földes, Rejtő, and Winter Kernel Estimation
BSKE	Blum and Susarla Kernel Estimation
KLME	Klein, Lee and Moeschberger Partially Parametric Estimation
SP-KME	Semi-Parametric KME

the nonparametric estimators are location invariant provided the censoring distribution makes the same location shift, therefore, the MISE for the nonparametric methods is the same regardless of whether a location parameter must be estimated or not. As expected, however, the MISE for the Weibull distribution when using parametric estimation is slightly higher when a location parameter must be estimated, but is still better than the nonparametric methods when the underlying family

Table 8 MISE: Exponential with Exponential Censoring.
(Standard deviation shown in parentheses)

		Parametric	*Nonparametric*							*Semi-parametric*	
		MLE	KME	TS-KME	TSJ-KME	MONE	PEXE	FRWE	BSKE	KLME	SP-KME
$n = 20$	$q = 0.25$	0.8112	2.0268	1.5961	1.8483	1.9543	1.5224	1.4271	3.0397	1.3419	1.6006
		(1.1364)	(1.3024)	(1.1798)	(1.4420)	(1.6678)	(1.4939)	(1.1459)	(4.6282)	(1.2950)	(1.5851)
	$q = 0.50$	1.3766	3.4437	3.6083	2.7744	3.7773	2.8737	2.5368	15.9050	2.1773	2.7260
		(2.5236)	(1.7666)	(2.3310)	(1.7097)	(2.5364)	(4.7216)	(1.6752)	(21.685)	(3.7839)	(4.2203)
	$q = 0.75$	4.5260	7.9221	11.0654	6.4419	10.0685	5.0602	6.8436	78.4118	8.3337	8.7649
		(2.6996)	(2.3513)	(4.5127)	(2.4521)	(4.4099)	(6.0409)	(2.6091)	(67.570)	(28.697)	(27.988)
$n = 60$	$q = 0.25$	0.2848	0.6737	0.5764	0.6273	0.6572	0.5497	0.5340	1.0294	0.4541	0.5837
		(0.4227)	(0.4478)	(0.4250)	(0.4625)	(0.5208)	(0.4634)	(0.4184)	(1.4378)	(0.4562)	(0.5244)
	$q = 0.50$	0.4502	1.4503	1.4196	1.1794	1.5558	1.1458	1.1371	6.9644	0.7115	1.0530
		(0.6839)	(0.7820)	(0.9007)	(0.7710)	(1.1889)	(1.5118)	(0.7253)	(8.7557)	(0.9479)	(1.3078)
	$q = 0.75$	1.0045	4.8343	6.6710	3.8708	6.0557	3.3489	4.2272	53.5711	2.4709	2.9269
		(2.4909)	(1.4290)	(2.7796)	(1.5080)	(2.6970)	(4.3274)	(1.4607)	(46.537)	(5.6839)	(6.0372)

Table 9 MISE: 2-Parameter Weibull, shape 2, with Exponential Censoring.
(Standard deviation shown in parentheses)

		Parametric	*Nonparametric*							*Semi-parametric*	
		MLE	KME	TS-KME	TSJ-KME	MONE	PEXE	FRWE	BSKE	KLME	SP-KME
$n = 20$	$q = 0.25$	0.5374	1.0362	0.7772	1.1414	0.8691	0.7312	0.6525	1.1738	0.6911	0.8014
		(0.5899)	(0.8034)	(0.6565)	(0.9573)	(0.6962)	(0.6316)	(0.6400)	(1.4169)	(0.5847)	(0.6146)
	$q = 0.50$	0.8353	1.6501	1.3414	1.5575	1.5952	1.3348	0.9503	5.1860	0.9285	1.2305
		(1.0528)	(1.1619)	(1.1679)	(1.2258)	(1.5747)	(2.1162)	(0.9192)	(8.0251)	(1.0255)	(1.2057)
	$q = 0.75$	1.6297	3.6714	4.4250	2.9587	4.1457	4.0261	2.2049	26.7995	1.6210	2.2470
		(2.1278)	(2.1618)	(3.7516)	(1.9630)	(3.4197)	(7.2607)	(1.7738)	(26.494)	(1.9378)	(2.5209)
$n = 60$	$q = 0.25$	0.1750	0.3120	0.2694	0.3239	0.2870	0.2635	0.2280	0.3020	0.2320	0.2782
		(0.2035)	(0.2312)	(0.2150)	(0.2626)	(0.2345)	(0.2182)	(0.2098)	(0.2810)	(0.2106)	(0.2244)
	$q = 0.50$	0.2601	0.5169	0.4238	0.4816	0.4930	0.4097	0.3521	1.2848	0.2960	0.4272
		(0.3195)	(0.3805)	(0.3349)	(0.3672)	(0.4046)	(0.3666)	(0.3422)	(1.5940)	(0.3185)	(0.3737)
	$q = 0.75$	0.6352	1.4160	1.3799	1.1466	1.4965	1.3375	0.9546	12.1120	0.6419	1.0189
		(0.9155)	(0.8854)	(1.1243)	(0.7908)	(1.1604)	(2.5702)	(0.7999)	(13.240)	(0.9139)	(1.2686)

Table 10 MISE: 2-Parameter Weibull, shape 3.5, with Exponential Censoring.
(Standard deviation shown in parentheses)

		Parametric	*Nonparametric*							*Semi-parametric*	
		MLE	KME	TS-KME	TSJ-KME	MONE	PEXE	FRWE	BSKE	KLME	SP-KME
$n = 20$	$q = 0.25$	0.3416	0.6785	0.4986	0.8549	0.5177	0.4560	0.4777	0.6354	0.4416	0.5053
		(0.3813)	(0.5592)	(0.4292)	(0.7234)	(0.3949)	(0.3884)	(0.5033)	(0.8828)	(0.3782)	(0.3895)
	$q = 0.50$	0.5162	1.0357	0.7709	1.1736	0.8848	0.6964	0.6634	2.6802	0.5770	0.7489
		(0.5682)	(0.8314)	(0.7019)	(0.9823)	(0.7650)	(0.6989)	(0.7018)	(4.2215)	(0.5570)	(0.6198)
	$q = 0.75$	0.9311	2.2821	2.2763	2.1361	2.1626	2.3414	1.4700	14.1739	0.9605	1.3716
		(1.0866)	(1.7682)	(2.4945)	(1.6866)	(2.0714)	(1.5430)	(1.4506)	(15.740)	(1.0811)	(1.2432)
$n = 60$	$q = 0.25$	0.1084	0.1926	0.1664	0.2091	0.1725	0.1618	0.1447	0.1730	0.1453	0.1716
		(0.1266)	(0.1512)	(0.1319)	(0.1776)	(0.1333)	(0.1290)	(0.1427)	(0.1471)	(0.1304)	(0.1371)
	$q = 0.50$	0.1558	0.3006	0.2426	0.3014	0.2681	0.2340	0.1970	0.5317	0.1810	0.2534
		(0.1875)	(0.2475)	(0.1950)	(0.2519)	(0.2109)	(0.1907)	(0.2151)	(0.5183)	(0.1881)	(0.2082)
	$q = 0.75$	0.3342	0.7040	0.5569	0.6502	0.6464	0.5246	0.4175	4.3526	0.3463	0.5399
		(0.4310)	(0.5328)	(0.4802)	(0.5343)	(0.5673)	(0.6072)	(0.4464)	(6.0305)	(0.4303)	(0.5223)

of the distribution is known. This brings up the important point that the ML estimation has the advantage of the assumption of a correctly specified model in this comparison.

Of the nonparametric estimation methods, the estimator proposed by Földes, Rejtő, and Winter is the best with the PEXE not far behind according to the MISE criterion. The KME is the next best and the easiest to compute while the estimator of Blum and Susarla is the worst, particularly for heavier censoring.

The semi-parametric methods fall between the nonparametric and parametric methods in terms of MISE. They do, however, have the slight advantage over the nonparametric estimators in this study because the Weibull distribution was used for the parametric components, which happened to be the distribution used in this comparison. The Klein, Lee and Moeschberger estimator, for example, behaves more and more like the ML estimator with increasing amounts of censoring. The MISE was not computed for the semi parametric estimators for the cases using the 3-parameter Weibull, but the estimation of the location parameter would be necessary and would most likely result in slight increases in MISE.

For further comparison, the Monte Carlo samples of 1000 ISE measures were obtained for each of the 3 distributions under consideration for samples of size 20 and 60 at each level of expected censoring. That is, the 10 estimation techniques were compared under 18 different conditions. This amounts to a very large number of comparisons when taking into account all of the pairwise comparisons as well as comparisons using various subsets of the 10 estimators under each scenario. A great deal of these subsets and pairwise comparisons were examined using the Kruskal-Wallis test to try to determine which estimators could be considered better and which ones could be considered as equal for the sample sizes and proportions of censoring considered in this study. The comparison criterion here was not MISE but results from Kruskal-Wallis tests. The Kruskal-Wallis test makes no assumptions on the underlying distribution of the ISE while the ISE comparison via the two-number summary of mean and standard deviation implies a more symmetric underlying distribution, which

the ISE is not. Although the results from all of the Kruskal-Wallis tests performed are not printed in this document, the overall summary of the comparison of the 10 estimators based on the results of many Kruskal-Wallis tests and many side-by-side boxplot comparisons of ISE is as follows. When the underlying distribution is correctly specified, maximum likelihood is significantly better than any of the other estimators in all cases examined here. In most cases, especially when expected censoring is at 50% or 75%, the kernel estimator of Blum and Susarla is significantly worse than all of the other estimators. The reason for this is its inability to extrapolate beyond the last observed failure time and is illustrated in figures 25, 26, and 27 in Appendix C. The semi-parametric Kaplan-Meier estimator and the partially parametric Klein, Lee, and Moeschberger estimator performed competitively, especially the KLME, but did enjoy the advantage of having a correctly specified parametric part. In fact, in a data set in which there is 100% censoring the KLME is identical to the MLE.

Now, when comparing the strictly nonparametric estimators two of them stood out above the rest: the piecewise exponential estimator (PEXE) and the kernel estimator proposed by Földes, Rejtő, and Winter (FRWE). The PEXE and the FRWE performed significantly better than the three Kaplan-Meier-based estimators and the mean order number estimator (MONE) in nearly all of the comparisons. Furthermore, in a head-to-head comparison the PEXE and the FRWE were statistically equivalent for the Weibull distributions at sample size $n = 20$ regardless of the amount of censoring. However, the PEXE had significantly lower ISE when the underlying distribution was exponential whereas the FRWE had a significantly lower ISE than the PEXE for both of the Weibull distributions at $n = 60$ for each level of censoring. In general, the MONE was not significantly different from the three Kaplan-Meier-based estimators with the exception of the cases when expected censoring was 75%, where it was significantly worse. The three Kaplan-Meier-based estimators performed similarly under most of the conditions. The trigonometric smoothing offered no significant improvement in ISE of the Kaplan-Meier estimator and the jackknifing procedure actually increased ISE at 25% censoring for each distribution and tended to have higher ISE for

the symmetric distribution. It is also interesting to note that all of the estimators are better in the sense of ISE at estimating symmetric distributions, such as the Weibull with shape 3.5, than skewed ones like the exponential.

III. *Goodness-of-Fit*

3.1 Literature Review

3.1.1 Tests of Simple Hypothesis. Many of the procedures found in the literature are for testing fit to a completely specified distribution function. In 1976, Koziol and Green [81] derived Cramér-von Mises type statistics using the Kaplan-Meier product limit estimate of the EDF for randomly censored samples in which the censoring survivor function, say H, depends on the failure survivor function, F, in the following way: $H = F^b$, where b is a censoring parameter between 0 and 2. The proportion of censoring is given by $q = \frac{b}{b+1}$. The statistics are used to test goodness-of-fit to a completely specified distribution function. They present asymptotic percentage points as well as a power study against certain alternatives at various levels of censoring. Csörgő and Horváth [31] consider using the Efron transformation [43] to convert the Koziol-Green statistics into one with a different asymptotic distribution. Hyde [69] used martingale theory in 1977 to show asymptotic normality of statistics used for testing the fit of a randomly censored sample to a completely specified distribution function for both discrete and continuous random variables. In 1979 Hollander and Proschan [66] developed a test statistic to test whether an underlying distribution subject to random censoring is a completely specified distribution and compared it to both the Koziol-Green statistic and Hyde's procedure. Ebrahimi and Habibullah [42] modified the Hollander and Proschan [66] statistic in 1992 to enhance its effectiveness for instances when the failure distribution and censoring distribution are proportionally related (as in the Koziol-Green model).

Turnbull and Weiss [136] proposed a likelihood ratio statistic in 1978 for testing goodness-of-fit with grouped data subject to random right censoring. Similarly, Gail and Ware [51] in 1979 presented a statistic for comparing randomly right censored data that is grouped to a completely specified distribution function or a known comparison curve derived from actuarial tables. Another test for grouped data was given in 1984 by O'Neill [104]. Versions of Kolmogorov-Smirnov, Kuiper,

and Cramér-von Mises statistics were introduced in 1980 by Koziol [80] for testing goodness-of-fit with randomly censored data when the distribution in the null hypothesis is completely specified. In this paper, Koziol provided a power study in addition to Monte Carlo analysis of the adequacy of the asymptotic distributions in finite samples. Burke [15] derived a test statistic for testing exponentiality with randomly censored samples for a simple hypothesis in 1980 as well. The Kolmogorov-Smirnov statistic was modified by Fleming, O'Fallon, O'Brien, and Harrington [46] to test goodness-of-fit of a completely specified distribution in the presence of random censoring. Hollander and Peña in 1992 [65] developed a chi-squared goodness-of-fit test for randomly censored data.

3.1.2 Tests of Composite Hypothesis. Relatively few goodness-of-fit tests have been developed for randomly right-censored data in the composite hypothesis case when parameters are estimated. Previously published tests are displayed in Table 11. The first procedure was a chi-squared test presented by J. Chen in a Ph. D. dissertation for Oregon State University in 1975 [22]. Another version of a chi-squared test for randomly censored data was developed by Habib [58], also in a 1981 doctoral dissertation for Oregon State University, which was later published by Habib and Thomas in 1986 [57]. Other variations of the chi-squared test for the composite hypothesis with randomly censored data are given by Akritas in 1988 [3] and J. H. Kim in 1993, each of which can also be used to test simple hypotheses as well. C. H. Chen [20,21] derived a correlation-based statistic in a 1982 dissertation, a generalization of the Shapiro-Francia statistic, for testing goodness-of-fit with randomly censored data when a scale parameter or both location and scale parameters are unknown.

In 1981 Nair [103] proposed two classes of nonparametric, large-sample tests of fit based on the maximum and average weighted difference between the specified survival function and the Kaplan-Meier estimate of the true survival function.

Table 11 Previous Goodness-of-Fit Tests for Randomly Censored Data with Composite H_0.

Year	Author(s)	Type	Comments
1975	J. Chen	Chi-Squared	
1980	M. Burke	Modified EDF	Applied to exponential failure with exponential censoring.
1982	C. H. Chen	Correlation	Applied to exponential failure with various censoring distributions including the exponential.
1986	Habib & Thomas	Chi-Squared	Derived asymptotic distribution of the stochastic process $Z(t) = n^{\frac{1}{2}}[\hat{F}_n(t) - F_T(t;\hat{\Theta}_n)]$
1988	M. Akritas	Chi-Squared	
1993	J.H. Kim	Chi-Squared	

A test for the exponential distribution when the scale parameter is estimated was derived by Burke in 1980 [15]. More specifically, Burke considers the case of an exponential lifetime distribution censored randomly on the right by an exponentially distributed censoring distribution. Making use of the Efron transformation, Burke's Cramér-von Mises-type test statistic is constructed in such a way that its asymptotic distribution is exactly the same as the asymptotic distribution of the Cramér-von Mises statistic for complete samples when the scale parameter is estimated regardless of the degree of censoring. However, simulations indicate that the small sample properties of Burke's test statistic will depend on the level of censoring as well as sample size and are not the same as the Cramér-von Mises statistic in the uncensored case. No small sample studies of Burke's statistic have been published and so consequently no small sample percentage points are available for goodness-of-fit testing. Percentage points were generated through Monte Carlo simulation for Burke's statistic for sample sizes 20(20)100 and proportions of censoring 0.20, 0.50, and 0.80 and are tabled in Section 3.5.2. Monte Carlo samples of size 250,000 were used.

3.2 Asymptotic Distributions of KME-Modified Test Statistics

3.2.1 Literature Review of General Asymptotic Theory. The function $Z(t) = \sqrt{n}[\hat{F}_n(t) - F_T(t)]$ represents a stochastic process in t and can be used to construct a variety of goodness-of-fit test statistics, including the Anderson-Darling and the Cramér-von Mises statistics. Efron [43: page 843] mentions the limiting normality of this stochastic process, namely that as n approaches infinity,

$Z(t)$ approaches a Gaussian stochastic process with mean 0 and covariance function

$$cov(Z(s), Z(t)) = [1 - F_T(s)][1 - F_T(t)] \int_{-\infty}^{s} \frac{dF_T(u)}{[1 - F_C(u)][1 - F_T(u)]^2} \tag{14}$$

for $s \leq t$. A proof of this convergence is given in Breslow and Crowley [13] though Burke contends that the proof is incomplete [16]. Let us assume now that we are dealing only with positive random variables. Efron [43], using a transformation from Doob [36], states that

$$Y(t) = \frac{n^{\frac{1}{2}}\{\hat{F}_n\left(a^{-1}(t)\right) - F_T\left(a^{-1}(t)\right)\}}{1 - F_T\left(a^{-1}(t)\right)}$$

is a stochastic process whose limit is a standard Weiner process on $0 \leq s < t \leq T$ where $a^{-1}(t)$ is the inverse function of

$$a(t) = \int_0^s \frac{dF_T(u)}{[1 - F_C(u)][1 - F_T(u)]^2}$$

and T is any value such that $F_T(T) > 0$. A standard Weiner process is a Gaussian process with mean 0 and $cov(s, t) = \min\{s, t\}$, also commonly known as Brownian motion.

Looking more closely at Doob's transformation [36] affords a little insight into the formulation of $a(t)$ and $a^{-1}(t)$. The purpose of the transformation is to convert a general Gaussian process into a more manageable Brownian motion, or standard Weiner, process. According to Doob [36] this can be achieved if the covariance function of the Gaussian process has the form $cov(s, t) = u(s)v(t), s < t$, for s and t in some interval and if the ratio $a(t) = \frac{u(t)}{v(t)}$ is continuous and monotone increasing with inverse function $a^{-1}(t)$. If these conditions are met, then the process $\frac{u[a^{-1}(t)]}{v[a^{-1}(t)]}$ is a standard Weiner process. For Efron's transformation, let

$$u(s) = F_T(s) \int_0^s \frac{dF_T(u)}{[1 - F_C(u)][1 - F_T(u)]^2}$$

and $v(t) = 1 - F_T(t)$. Thus,

$$a(t) = \frac{u(t)}{v(t)} = \int_0^t \frac{dF_T(u)}{[1 - F_C(u)][1 - F_T(u)]^2}.$$

Csörgő and Horváth [31] look more closely at the Efron transform within the Koziol-Green (proportional hazards) model of random censorship for a simple hypothesis. Under this model, they show that $a^{-1}(t)$ can be estimated by

$$a_n^{-1}(t) = G^0\left(1 - \left(\frac{t}{\hat{p}} + 1\right)^{-\hat{p}}\right)$$

where G^0 is the inverse of the CDF F^0 specified by the null hypothesis and $\hat{p} = \frac{r}{n}$ is the observed proportion of failures in the sample. Csörgő and Horváth's Efron-transformed process [31] is given by

$$Y_n^0(t) = \frac{n^{\frac{1}{2}}\{\hat{F}_n^0[a_n^{-1}(t)] - F^0[a_n^{-1}(t)]\}}{1 - F^0[a_n^{-1}(t)]}$$

and corresponding Cramér-von Mises statistic for lifetime distributions bounded on the left by zero, as many lifetime distributons are, may be written as

$$\omega_n^2 = \int_0^T [Y_n^0(t)]^2 dt$$

where T is chosen so that $a_n^{-1}(T) = x_{(n)}$. Now, as a result of the Efron transform, $Y_n^0(t)$ is a Brownian motion process asymptotically and by the scale transformation of Brownian motion [31]

$$\omega_n^2(0, T) = T^2 \omega_n^2(0, 1)$$

in distribution. This scale transformation identifies the process as the integral of the square of a Brownian Bridge process, which was derived by Cameron and Martin in 1944 [31] and is tabled in Csörgő and Horváth [31]. The Brownian Bridge process is Brownian motion on the interval (0,1).

So far we have only discussed the asymptotic distribution of test statistics for simple hypotheses. In 1986, Habib and Thomas [57] derived the limiting distribution of the stochastic process

$$\hat{Z}_n(t) = n^{\frac{1}{2}}[\hat{F}_n(t) - F_T(t, \hat{\theta}_n)]$$

where θ is a vector of k parameters to be estimated by maximum likelihood to be used for goodness-of-fit with composite hypotheses. They [57] determined the limiting distribution to be Gaussian with zero mean and covariance given by

$$cov[\hat{Z}(s), \hat{Z}(t)] = cov[Z(s), Z(t)] - \frac{\partial F_T(s;\theta)}{\partial \theta}^T J^{-1} \frac{\partial F_T(t;\theta)}{\partial \theta} \tag{15}$$

where T represents transpose and J denotes the information matrix whose elements are given by

$$J_{ij} = -\int \frac{\partial^2 \ln f_T(t,\theta)}{\partial \theta_i \partial \theta_j} F_C(t) f_T(t;\theta) dt - \int \frac{\partial^2 \ln F_T(t,\theta)}{\partial \theta_i \partial \theta_j} F_T(t;\theta) f_C(t) dt \tag{16}$$

for $i, j = 1, \ldots, k$.

3.2.2 Problems with the Asymptotic Distribution of Test Statistics in Tests of Exponentiality with Exponential Censoring. For the composite test of hypothesis, consider an exponential lifetime distribution with mean θ subject to an exponential censoring distribution, which fits the proportional hazards criterion of the Koziol-Green model of random censorship. Let us also use the notation of Koziol and Green [81] to denote censoring parameter b, which is related to the proportion of uncensored observations by the relation $b = \frac{1-p}{p}$ and is easily estimated from the data. The asymptotic distribution of $\hat{Z}_n(t)$ should be Gaussian with zero mean and covariance

function

$$cov[\hat{Z}(s), \hat{Z}(t)] = \mathrm{e}^{-\frac{t}{\theta}} \left\{ \frac{1}{(b+1)} \exp\left[-\frac{s}{\theta} + \frac{(b+1)}{\theta} s\right] - \frac{1+(b+1)^2}{(b+1)} \exp\left[-\frac{s}{\theta}\right] \right\}$$

for $0 \leq s \leq t$ and $b \geq 0$, constructed using the methods given by Habib and Thomas.

Observe that the covariance function of the resulting Gaussian process can be factored into an expression in terms of t multiplied by an expression depending only on s. That is, let

$$v(t) = e^{-t/\theta}$$

and

$$u(s) = \frac{1}{(b+1)} \mathrm{e} \left[-\frac{s}{\theta} + \frac{(b+1)}{\theta} s\right] - \frac{1+(b+1)^2}{(b+1)} \exp\left[-\frac{s}{\theta}\right]$$

so $cov(\hat{Z}(s), \hat{Z}(t)) = u(s)v(t)$. Now, for the Doob/Efron transformation to Brownian motion, we need

$$\begin{aligned} a(t) &= \frac{u(t)}{v(t)} \\ &= \frac{1}{(b+1)} \exp\left[\frac{(b+1)}{\theta} t\right] - \frac{1+(b+1)^2}{(b+1)} \\ &= \frac{1}{(b+1)} \left\{ \exp\left[\frac{(b+1)}{\theta} t\right] - 1 - (b+1)^2 \right\}. \end{aligned}$$

Further noting that the proportion of uncensored observations is $p = \frac{1}{(b+1)}$, we have

$$a_n(t) = \hat{p} \exp\left[\frac{t}{\hat{p}\theta} - 1 - \frac{1}{\hat{p}^2}\right] \tag{17}$$

and thus,

$$a_n^{-1}(t) = \hat{p}\theta \ln\left(\frac{t}{\hat{p}} + 1 + \frac{1}{\hat{p}^2}\right) \tag{18}$$

where $\hat{p} = \frac{r}{n}$. When following this methodology to construct a goodness-of-fit test statistic whose asymptotic distribution is known, namely the integral of a squared Brownian Bridge in this case, the estimation of parameters interferes with the transformation. One problem is that $a_n^{-1}(0) > 0$. In other words, the transformation does not return time t to 0. This fact, in and of itself, is not a problem. However, a severe problem is that it is possible to get $a_n^{-1}(0) > X_{(n)}$, which means that $\hat{S}_n(a^{-1}(t)) = 0$ for $t > 0$ and so the Csörgő and Horváth test statistic $\omega_n^2(0,T)$ depends exclusively on the sample size. In other words, the test statistic does not contain enough information to acknowledge or rule out a hypothesized distribution and, furthermore, would be a constant for each given sample size regardless of the underlying or hypothesised distributions. In some instances it is also possible to get $a_n^{-1}(0) > T$, which is the upper limit of integration in the integral. As a result, there is no guarantee that the general Gaussian stochastic process $\hat{Z}_n(t)$ can be transformed into the Brownian motion process.

3.2.3 Problems with the Asymptotic Distribution of Test Statistics when Testing for the Weibull Distribution Within the Proportional Hazards Model of Random Censorship. Consider a goodness-of-fit test with a composite hypothesis in which the lifetime distribution is a 3-parameter Weibull distribution with known shape parameter. Further assume that the censoring model is the proportional hazards model of random censorship. That is, we have a lifetime distribution with distribution function

$$F_T(t;\gamma,\eta) = 1 - e^{-\left(\frac{t-\gamma}{\eta}\right)^\beta}$$

and a censoring distribution with distribution function

$$F_C(t;\gamma,\eta) = 1 - e^{-b\left(\frac{t-\gamma}{\eta}\right)^\beta}$$

where $t \geq \gamma, \eta > 0, \beta > 0, b > 0$ is the censoring parameter [81], and we assume that β is known. Applying Equations 14, 15, and 16 yields a covariance function for the zero mean Gaussian process

$\hat{Z}(t) = n^{\frac{1}{2}}[\hat{F}_n(t) - F_T(t, \hat{\theta}_n)]$ given by

$$\begin{aligned} cov(\hat{Z}(s), \hat{z}(t)) &= \frac{(s-\gamma)^{\beta-1}(t-\gamma)^{\beta-1}\mathrm{e}^{-\left(\frac{s-\gamma}{\eta}\right)^{\beta}-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}}{\Gamma(1-\frac{2}{\beta}) - \Gamma^2(1-\frac{1}{\beta})} \left\{ \frac{\beta^2(b+1)^{1-\frac{2}{\beta}}}{\eta^{2\beta-2}(\beta-1)^2} \right. \\ &\left. + \frac{\beta(b+1)^{1-\frac{1}{\beta}}\Gamma(1-\frac{1}{\beta})}{\eta^{2\beta-1}(\beta-1)}(s+t-2\gamma) + \frac{(b+1)\Gamma(1-\frac{2}{\beta})}{\eta^{2\beta}}(s-\gamma)(t-\gamma) \right\} \end{aligned}$$

when using the methods of Habib and Thomas. A regularity condition imposed by this function is that the shape parameter β must be greater than 2. Another result that surfaces is that, although this model has a censoring distribution with a hazard function that is proportional to the failure distribution, the covariance function takes on a form that cannot be factored as $cov(\hat{Z}(s), \hat{Z}(t)) = u(s)v(t)$, thus preventing the transformation to Brownian motion via the Doob/Efron transform.

3.2.4 Remarks on Asymptotic Distributions of Goodness-of-Fit Statistics. When data is randomly censored, distribution function estimators, parameter estimators, and goodness-of-fit test statistics converge slower. For example, while the EDF for an uncensored sample converges at a rate on the order of $n^{\frac{1}{2}}(\log n)^{-1}$, the KME converges at a rate on the order of $n^{\frac{1}{4}}(\log n)^{-\frac{1}{2}}$ [48] and the kernel estimator given by Földes, Rejtő, and Winter converges at a rate on the order of $n^{\frac{1}{8}}(\log n)^{-\frac{1}{4}}$ [49]. In the construction of confidence bands for the Kaplan-Meier estimator it is stated [16, 31] that a sample size of at least $n = 81 \times 10^6$ is required for 89% confidence using the methods of Gillespie and Fisher [55]. Csörgő and Horváth, however, were able to develop a methodology for constructing such confidence bands which they say require samples of size $n = 35,000$, which is still quite large [31].

It also seems intuitive that the greater the amount of censoring, the slower the convergence. The following passage from Silverman [120: pages 73-74] was in reference to the research of Bickel and Rosenblatt [10] on the asymptotic behavior of kernel density estimators for uncensored data but also has relevance here.

> This discussion is not intended to belittle their remarkable mathematical achievement. The authors themselves point out that 'these asymptotic calculations are to be taken

> with a grain of salt' and Rosenblatt warns in his survey(1971, p.1818) against the over-literal interpretation of asymptotic results. Nevertheless, he goes on to say, asymptotic theorems are useful if treated with care. For example, they can be used as a starting-point for simulation studies or simulation-based procedures, and they may help to give an intuitive feel for the way that a method will behave in practice.

Currently there are goodness-of-fit tests for randomly censored data for which the asymptotic behavior of percentage points is established. However, no tables of percentage points have been published for small samples. Since such tables can only be constructed through Monte Carlo simulation, this is one area where the applied statistician can make a difference.

3.3 Some Justification for the Assumption of an Exponentially Distributed Censoring Variable

Because of complex dependencies of the distributions of goodness-of-fit statistics on the censoring distribution, it is somewhat necessary to assume some kind of model for censoring. For distributions which are bounded on the left, consider a censoring distribution that is negative exponential with the same location parameter as the lifetime distribution. Many distributions commonly used in reliability theory share this characteristic, such as the Weibull, Gamma, and lognormal. Some justification for this assumption may be afforded by a 1960 paper by R. F. Drenick [37], who, generalizing the work of Palm [110], provides proof that systems composed of many components in a series arrangement tend to have exponentially distributed lifetimes under reasonably general conditions as the complexity of the system and the operation time increase. In order for Drenick's Theorem to have some applicability in this random censoring model, we must assume that all modes of censoring are independent, any means of censoring will remove a subject or item from the test, and each mode of censoring always exists. With these assumptions, the censoring mechanisms for a randomly censored sample may be viewed as a complex series system since there may be several modes of censorship for each subject or item on test, any one of which will result in withdrawal from the test. Hence, the assumption of an exponentially distributed censoring variable may be reasonable and somewhat robust.

3.4 Modified EDF Statistics for New Goodness-of-Fit Tests for Randomly Censored Data

Goodness-of fit statistics based on the empirical distribution function, or EDF statistics, are known to be generally superior to other classes of goodness-of-fit tests in terms of power when it comes to detecting departures from a variety of continuous univariate distributions [32, 123]. The most successful of which is the Anderson-Darling statistic followed closely by the Cramér-von Mises statistic. It seems reasonable to assume that they may enjoy the same status among goodness-of-fit test statistics for randomly censored samples.

3.4.1 Computing Formulas. For complete samples under a simple hypothesis, the modified Cramér-von Mises and Anderson-Darling statistics were defined by Equations 1 and 2, respectively, in Chapter II. In the case of the composite hypothesis $F_0(x)$ may be replaced by its maximum likelihood estimate. Further in Chapter II it is shown that these statistics may be modified by using the Kaplan-Meier estimator in place of the EDF and employed as the measure that is minimized for distance estimation when samples are randomly right-censored. Although these test statistics work well as distance measures within the context of distance estimation, a problem arises when they are used for goodness-of-fit testing.

The problem, which is prevalent primarily when data are subject to censoring levels above 50%, stems from the fact that estimators of the EDF do a very poor job after the last observation in the sample. For highly censored data, this will often yield a higher test statistic for the correct distribution than for some alternative distribution in a goodness-of-fit test. Figure 2 is helpful in seeing how the KME, for example, is fairly accurate for a correctly specified null distribution up to the last observation, which is circled in the figure, but then is quite far off because it takes on the value 1 beyond $X_{(n)}$. Now, when the null distribution is incorrect the KME, represented by the dash-dot line in Figure 2, is not as good from 0 to $X_{(n)}$ but does well enough from $X_{(n)}$ to $+\infty$ to more than make up for the discrepancies that are measured from 0 to $X_{(n)}$, resulting in a lower test statistic value than what is computed when the null distribuion is correct. The effect of this

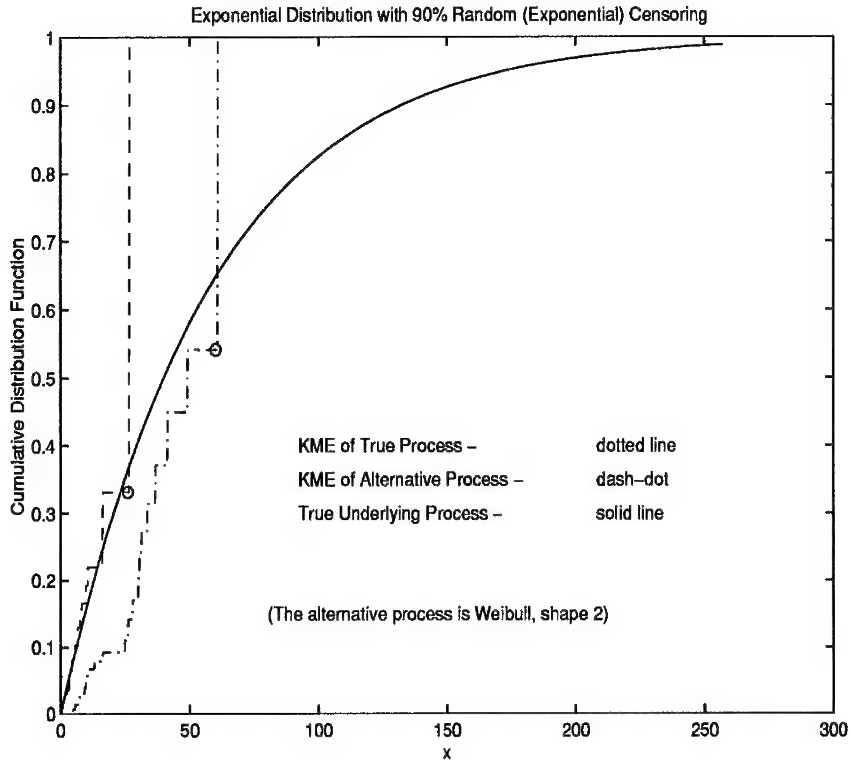

Figure 2 Illustration of Upper Integration Limit for the KME-Modified Cramér-von Mises Statistic.

poor ability to estimate beyond $X_{(n)}$ is that for highly censored samples it would be more likely to reject a true null hypothesis than a false one. This problem is corrected simply by replacing integration limit of $+\infty$ with $X_{(n)}$ when computing the modified goodness-of-fit statistics. Upon making this slight change, the resulting computing formulas for the KME-modified Cramér-von Mises and Anderson-Darling test statistics are

$$W_{r,n}^2 = n\sum_{j=1}^{r}\{[\hat{F}_n(x_{(j-1)})]^2(U_{(j)} - U_{(j-1)}) - \hat{F}_n(x_{(j-1)})(U_{(j)}^2 - U_{(j-1)}^2) + \frac{1}{3}(U_{(j)}^3 - U_{(j-1)}^3)\} \quad (19)$$

and

$$\begin{aligned} A_{r,n}^2 &= n\sum_{j=1}^{r}\{[\hat{F}_n(x_{(j-1)})]^2(\log U_{(j)} - \log U_{(j-1)}) \\ &\quad - [\hat{F}_n(x_{(j)}) - 1]^2[\log(1 - U_{(j)}) - \log(1 - U_{(j-1)})] - (U_{(j)} - U_{(j-1)})\} \end{aligned} \quad (20)$$

respectively, where $\hat{F}_n(X_{(0)}) = U_0 = 0$.

3.4.2 A Condition for Location and Scale Invariance. The maximum likelihood estimators for randomly censored data are location and scale invariant provided the censoring distribution has the same location parameter as the lifetime distribution and makes the same location and scale changes as the lifetime distribution. The KME is always location invariant when the censoring distribution makes the same location and scale changes as the lifetime distribution but is only scale invariant under these transformations when the location parameters are equal. This can be seen empirically by using the same set of uniform random numbers to generate randomly censored samples from distribution pairs with location and/or scale changes. To get an identical product-limit estimate it is necessary to have the exact same series of zeros and ones for the covariate indicator function δ_i, which only occurs when the location parameters are equal and the location and the scale of the censoring distribution undergo the same transformations as the underlying failure distribution.

Some analytical underpinnings for this are shown in the following argument. Let $u_j, j = 1, \ldots, 2n$ be a sequence of random numbers from the uniform distribution on $[0, 1]$. Then random deviates from failure, $F_T\left(\frac{t-\gamma_1}{\theta_1}\right)$, and censoring, $F_C\left(\frac{c-\gamma_2}{\theta_2}\right)$, distributions with location and scale parameters γ_i and $\theta_i > 0, i = 1, 2$, respectively, may be generated using probability integral transformations of the form

$$t_i = \theta_1 F_T^{-1}(u_j) + \gamma_1, \quad i = 1, \ldots, n, j = 1, \ldots, n$$

$$c_i = \theta_2 F_C^{-1}(u_j) + \gamma_2, \quad i = 1, \ldots, n, j = n+1, \ldots, 2n.$$

The observed data pair then consists of $x_i = \min\{t_i, c_i\}$ and $\delta_i = I_{[t_i \le c_i]}$ in the competing risks model of random censorship. Now, suppose we let $s > 0$ denote a change in scale and h denote a location shift. This would change the probability integral transformations in the following way

$$t_i' = s\theta_1 F_T^{-1}(u_j) + \gamma_1 + h, \quad i = 1, \ldots, n, j = 1, \ldots, n$$

$$c_i' = s\theta_2 F_C^{-1}(u_j) + \gamma_2 + h, \quad i = 1, \ldots, n, j = n+1, \ldots, 2n.$$

Suppose $t_i \le c_i$, which impies that $\delta_i = 1$. Then we have

$$\theta_1 F_T^{-1}(u_j) + \gamma_1 \le \theta_2 F_C^{-1}(u_j) + \gamma_2.$$

Now if δ_i is to remain at 1 after the location and scale shift we need $t_i' \le c_i'$, or

$$s\theta_1 F_T^{-1}(u_j) + \gamma_1 + h \le s\theta_2 F_C^{-1}(u_j) + \gamma_2 + h.$$

Furthermore, it can be seen from this that if $\gamma_1 = \gamma_2$ then we can be assured that

$$\theta_1 F_T^{-1}(u_j) \le \theta_2 F_C^{-1}(u_j)$$

since $s > 0$. However, if $\gamma_1 \ne \gamma_2$, then there is no guarantee that $t_i' \le c_i'$ and many counterexamples can easily be constructed.

Since the modified Cramér-von Mises and Anderson-Darling goodness-of-fit statistics given in this dissertation are based on maximum likelihood and Kaplan-Meier estimates, which are location and scale invariant under the condition that the location parameters are equal, then it follows that they are location and scale invariant under this condition as well. It should also be noted that the minimum distance estimation of location parameters is also location invariant.

3.5 New Goodness-of-Fit Tests for Exponential Lifetimes with Exponentially Distributed Random Right Censoring for a Composite Hypothesis

3.5.1 New Tests Based on KME-Modified Cramér-von Mises and Anderson-Darling Test Statistics. Monte Carlo samples of size 250.000 were used to find percentage points for sample sizes 20(20)200 and proportions of censoring 0.10(0.10)0.90 for the two new goodness-of-fit tests for exponential lifetimes subject to exponentially distributed random right-censoring based on the

modified Cramér-von Mises and Anderson-Darling statistics $W^2_{r,n}$ and $A^2_{r,n}$. The statistics are modified with the natural choice of the Kaplan-Meier estimator replacing the EDF. These tests are based on the competing risks model of random censorship and the assumption that the failure and censoring distributions are independent and both exponentially distributed. The null hypothesis is $\mathrm{H}_0 : F_T(t)$ is exponential. The procedure for testing this hypothesis is as follows:

1. Estimate λ using the maximum likelihood estimator $\hat{\lambda} = \frac{1}{r}\sum_{i=1}^{n} x_i$ where $r = \sum_{i=1}^{n} \delta_i$.
2. Construct the KME as defined in Section 2.2.1.
3. Calculate the test statistic, either $W^2_{r,n}$ or $A^2_{r,n}$, using Equation 19 or 20.
4. Estimate the proportion of censoring as $\hat{q} = 1 - \frac{r}{n}$.
5. Enter the appropriate table of percentage points for the nearest n and q at the desired α level and compare the test statistic to the corresponding percentage point.
6. Reject H_0 if the test statistic is greater than the percentage point from the table, otherwise do not reject the hypothesis of exponentiality.

Tables of percentage points are given in Appendix D.

3.5.2 Burke's Test for Exponentiality for a Composite Hypothesis. For Burke's test of exponentiality, we assume that the failure and censoring random variables are independent and exponentially distributed with $F_T(t) = 1 - \mathrm{e}^{-t/\lambda}$ and $F_C(c) = 1 - \mathrm{e}^{-c/\theta}$. Though Burke does not explicitly give his test statistic a name, we will call it B^2. It can be expressed as

$$B^2 = \int_0^{\hat{M}_n} \left(\frac{n}{r}\right)^2 \left(\mathrm{e}^{-\frac{t}{\hat{\theta}}}\right)^2 \left(\frac{1}{\hat{\theta}} + \frac{1}{\hat{\lambda}}\right) \mathrm{e}^{-t\left(\frac{1}{\hat{\theta}}+\frac{1}{\hat{\lambda}}\right)} n \left[\hat{F}_n(t) - \mathrm{e}^{-\frac{t}{\hat{\lambda}}}\right]^2 dt \tag{21}$$

where $\hat{M}_n = \bar{x}\varepsilon \log n$ for some $\varepsilon > 0$, $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$, $\hat{\lambda} = \frac{1}{r}\sum_{i=1}^n x_i$, and $\hat{\theta} = \frac{1}{n}\left(\frac{r}{n-r}\right)\sum_{i=1}^n x_i$. Simpson's rule with 101 function evaluations was the numerical integration technique used and $\varepsilon = 1$ was selected for the construction of the table of small sample percentage points. See Appendix A for a description of Simpson's rule. Monte Carlo samples of size 250,000 were used to construct Table 12.

The null hypothesis is $\mathrm{H}_0 : F_T(t)$ is exponential. The procedure for testing this hypothesis is as follows:

1. Estimate λ using the maximum likelihood estimator $\hat{\lambda} = \frac{1}{r}\sum_{i=1}^n x_i$.
2. Estimate θ using the estimator $\hat{\theta} = \frac{1}{n}\left(\frac{r}{n-r}\right)\sum_{i=1}^n x_i$.
3. Construct the KME as defined in Section 2.2.1.
4. Calculate the test statistic B^2.
5. Estimate the proportion of censoring as $\hat{q} = 1 - \frac{r}{n}$.
6. Enter the appropriate table of percentage points for the nearest n and q at the desired α level and compare the test statistic to the corresponding percentage point.
7. Reject H_0 if the test statistic is greater than the percentage point from the table, otherwise do not reject the hypothesis of exponentiality.

Table 12 Percentage Points of B^2 for the Exponential Distribution with Exponential Censoring.

	n	α 0.25	0.20	0.15	0.10	0.05	0.025
$q = .20$	20	0.182	0.205	0.236	0.279	0.360	0.446
	40	0.160	0.180	0.206	0.245	0.314	0.387
	60	0.152	0.171	0.196	0.232	0.297	0.367
	80	0.148	0.166	0.190	0.225	0.289	0.355
	100	0.146	0.164	0.187	0.222	0.284	0.350
$q = .50$	20	0.199	0.229	0.270	0.332	0.450	0.578
	40	0.178	0.204	0.238	0.289	0.386	0.491
	60	0.170	0.195	0.227	0.276	0.365	0.460
	80	0.166	0.190	0.221	0.268	0.352	0.444
	100	0.165	0.188	0.218	0.264	0.347	0.437
$q = .80$	20	0.184	0.224	0.307	0.430	0.682	1.045
	40	0.164	0.201	0.252	0.356	0.539	0.764
	60	0.161	0.189	0.234	0.308	0.471	0.660
	80	0.160	0.187	0.228	0.294	0.437	0.605
	100	0.161	0.188	0.226	0.286	0.415	0.574

3.6 New Goodness-of-Fit Tests for the Weibull with Exponentially Distributed Random Right-Censoring for a Composite Hypothesis

3.6.1 New Tests Based on Modified Cramér-von Mises and Anderson-Darling Statistics for Unknown Location and Scale. Monte Carlo simulation used to find percentage points for sample sizes 20(20)100 and proportions of censoring 0.20, 0.50, and 0.80 for the two new goodness-of-fit tests for Weibull lifetimes with known shape parameter values $\beta = 2$ and $\beta = 3.5$ subject to exponentially distributed random right-censoring based on the Cramér-von Mises and Anderson-Darling statistics where the KME is substituted for the EDF. Monte Carlo samples of size 250,000 were used to find the percentage points, which are tabled in Appendix E. These tests are based on the competing risks model of random censorship and the assumption that the failure and censoring distributions are independent. The failure distribuition is assumed to be Weibull and the censoring distribution is exponential and they are assumed to be statistically independent. That is, we have a random sample of pairs $(x_1, \delta_1), \ldots, (x_n, \delta_n)$ where $x_i = \min\{t_i, c_i\}$ and $\delta_i = I_{[t_i \leq c_i]}$. The null hypothesis is $\mathrm{H}_0 : F_T(t)$ is Weibull with known shape parameter β. The procedure for testing this hypothesis is as follows:

1. Estimate the Weibull location via minimum distance estimation and scale parameter using maximum likelihood estimation as in Section 2.1.3 only with the shape parameter known.
2. Construct the KME as defined in Section 2.2.1.
3. Calculate the test statistic, either $W^2_{r,n}$ or $A^2_{r,n}$.
4. Estimate the proportion of censoring as $\hat{q} = 1 - \frac{r}{n}$.
5. Enter the table of percentage points for the Weibull with shape β under the nearest n and q at the desired α level and compare the test statistic to the corresponding percentage point.
6. Reject H_0 if the test statistic is greater than the percentage point from the table, otherwise do not reject the hypothesis that the failure process is Weibull with shape β.

Tables of percentage points to be used with these test are given in Appendix E.

3.7 New Goodness-of-Fit Tests Based on Crude Lifetimes

The competing risks model of random censorship is widely used because it makes sense to view the underlying failure process as one risk and the combination of every possible reason why an item may be withdrawn from a test as the other risk. In competing risks theory, a *net lifetime* represents the lifetime of an item when it is subject to one of the specified risks while no other risks are present and a *crude lifetime* represents the lifetime of an item subject to one of the specified risks when all risks are present [87: p. 110]. Our purpose is to characterize the distribution of the net lifetime for the risk of interest, which is our underlying failure process.

Let T represent the net lifetime of the variable of interest and let C characterize the net lifetime of a random variable that censors the observed life of T. In this context the observed lifetime of an item is $X = \min\{T, C\}$ as well as the covariate indicator δ. By convention, if the item is observed until failure, then $T \leq C$, $X = T$, and $\delta = 1$. If an item is withdrawn from testing before it fails or fails from any cause other than the cause of interest, then $T > C$, $X = C$, and

$\delta = 0$. We define the crude lifetimes as conditional lifetimes which depend on whether or not the observed life of an item on test represents a failure time or a withdrawal time. That is, let Y_T be the conditional life of X given that $X = T$ and let Y_C be the conditional life of X given that $X = C$. If we assume T and C to be independent, then

$$\begin{aligned} S_X(x) &= P[X > x] \\ &= P[T > x]P[C > x] \\ &= S_T(x)S_C(x). \end{aligned} \tag{22}$$

Furthermore, by conditioning,

$$\begin{aligned} S_X(x) &= P[X > x] \\ &= P[X = T]P[X > x \mid X = T] + P[X = C]P[X > x \mid X = C] \\ &= P[X = T]P[Y_T > x] + P[X = C]P[Y_C > x] \\ &= pS_{Y_T}(x) + (1-p)S_{Y_C}(x) \end{aligned} \tag{23}$$

where $p = P[X = T] = P[T \leq C]$. Thus, from Equations 22 and 23 we have the following relationship between the survivor functions of the observed, net, and crude lifetimes

$$S_X(x) = S_T(x)S_C(x) = pS_{Y_T}(x) + (1-p)S_{Y_C}(x). \tag{24}$$

This shows that the observed lifetime X can be modeled as both the minimum of competing net lifetimes as well as a mixture of crude lifetimes.

In practice, this means that from the observed random sample of $x_i = \min\{t_i, c_i\}, i = 1, \ldots, n$, with indicator $\delta_i = I_{[t_i \leq c_i]}$, the set of failure times $t_{(1)}, \ldots, t_{(r)}$ are the observed failures from a *randomly censored sample* of net lifetimes with distribution function $F_T(t)$ and can also be taken

as a *complete sample* of crude lifetimes with distribution function $F_{Y_T}(t)$. Furthermore, the set of withdrawal times $c_{(1)}, \ldots, c_{(n-r)}$ is a randomly censored sample of net lifetimes with distribution function $F_C(c)$ and can also be taken as a *complete sample* of crude lifetimes with distribution function $F_{Y_C}(c)$. The randomly censored sample can now be treated as a mixture of two distributions where the population from which each observation is drawn is known and the mixing parameter p is easily estimated by $\hat{p} = \frac{r}{n}$. Now, since the observed crude lifetimes are considered to be complete samples, we can use complete sample goodness-of-fit procedures to determine the distributions of the crude lifetimes of both the failure and the censoring variables. The incentive to fit a distribution to the crude lifetime of the censoring variable arises through the following relationship between crude lives and net lives. When net lifetimes are independent, the hazard function of the net life of the random variable T, our variable of interest, is

$$h_T(x) = \frac{p f_{Y_T}(x)}{p S_{Y_T}(x) + (1-p) S_{Y_C}(x)}. \tag{25}$$

A proof of this result is given in [87: pp. 288-290]. An alternative proof is given here as follows. Since Y_T is the conditional life of X given that $X = T$ and T and C are independent, the density function of Y_T can be expressed as

$$\begin{aligned} f_{Y_T}(x) &= \lim_{\Delta x \to 0} \frac{P[x \le Y_T \le x + \Delta x]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{P[x \le X \le x + \Delta x \mid X = T]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{P[x \le T \le x + \Delta x, X = T]}{\Delta x P[X = T]} \\ &= \lim_{\Delta x \to 0} \frac{P[x \le T \le x + \Delta x] P[C > x]}{\Delta x P[T \le C]} \\ &= \frac{f_T(x) S_C(x)}{p}. \end{aligned} \tag{26}$$

Solving Equation 26 for $f_T(x)$ yields

$$f_T(x) = \frac{pf_{Y_T}(x)}{S_C(x)}$$

which can then be written as

$$h_T(x)S_T(x) = \frac{pf_{Y_T}(x)}{S_C(x)}.$$

As a result, we have

$$\begin{aligned} h_T(x) &= \frac{pf_{Y_T}(x)}{S_T(x)S_C(x)} \\ &= \frac{pf_{Y_T}(x)}{pS_{Y_T}(x) + (1-p)S_{Y_C}(x)}. \end{aligned}$$

To further examine the intricate relationships between net and crude lifetimes, consider the survivor functions of the crude lifetimes expressed in terms of the net lifetimes. We have

$$\begin{aligned} S_{Y_T}(x) &= P[Y_T > x] \\ &= P[X > x \mid X = T] \\ &= P[T > x \mid T \leq C] \\ &= \frac{P[T > x, T \leq C]}{P[T \leq C]} \\ &= \frac{\int_x^\infty \left[\int_t^\infty f_T(t)f_C(c)dc\right] dt}{p} \\ &= \frac{\int_x^\infty f_T(t)S_C(t)dt}{p} \end{aligned}$$

and

$$\begin{aligned} S_{Y_C}(x) &= P[Y_C > x] \\ &= P[X > x \mid X = C] \end{aligned}$$

$$
\begin{aligned}
&= P[C > x \mid C < T] \\
&= \frac{P[C > x, C < T]}{P[C < T]} \\
&= \frac{\int_x^\infty \left[\int_c^\infty f_C(c) f_T(t)dt\right] dc}{1-p} \\
&= \frac{\int_x^\infty f_C(c) S_T(c) dc}{1-p}.
\end{aligned}
$$

By substituting the expressions we have obtained for $f_{Y_T}(x), S_{Y_T}(x)$, and $S_{Y_C}(x)$ into Equation 25, we gain insight into the identity

$$
\begin{aligned}
h_T(x) &= \frac{p f_{Y_T}(x)}{p S_{Y_T}(x) + (1-p) S_{Y_C}(x)} \\
&= \frac{p\left[\frac{f_T(x) S_C(x)}{p}\right]}{p\left[\frac{\int_x^\infty f_T(t) S_C(t) dt}{p}\right] + (1-p)\left[\frac{\int_x^\infty f_C(t) S_T(t) dt}{1-p}\right]} \\
&= \frac{f_T(x) S_C(x)}{\int_x^\infty f_T(t) S_C(t) dt + \int_x^\infty f_C(t) S_T(t) dt} \\
&= \frac{h_T(x) S_T(x) S_C(x)}{\int_x^\infty [h_T(t) S_T(t) S_C(t) + h_C(t) S_T(t) S_C(t)] dt} \\
&= \frac{h_T(x) S_T(x) S_C(x)}{\int_x^\infty [h_T(t) + h_C(t)] S_T(t) S_C(t) dt} \\
&= \frac{h_T(x) S_X(x)}{\int_x^\infty h_X(t) S_X(t) dt} \\
&= \frac{h_T(x) S_X(x)}{\int_x^\infty f_X(t) dt} \\
&= \frac{h_T(x) S_X(x)}{S_X(x)} \\
&= h_T(x).
\end{aligned}
$$

The goodness-of-fit test procedure can be posed as a simultaneous hypothesis test on the crude lifetimes of the failure and censoring variables. The null hypothesis is

$$
\begin{aligned}
\mathrm{H}_0: \quad & F_{Y_T} \in \{H_\phi, \phi = (\phi_1, \ldots, \phi_k)' \in \Phi\} \\
& F_{Y_C} \in \{G_\psi, \psi = (\psi_1, \ldots, \psi_l)' \in \Psi\}
\end{aligned}
$$

where H and G are families of distributions and ϕ and ψ are vectors of parameters in parameters spaces Φ and Ψ. The procedure for testing this hypothesis is as follows:

1. Estimate p by $\hat{p} = \frac{r}{n}$.
2. (a) Perform any complete sample goodness-of-fit test on the failure set.

   (b) Perform any complete sample goodness-of-fit test on the withdrawal set.
3. Use significance level $\frac{\alpha}{2}$ to achieve an overall level of significance of at most α.
4. Reject H_0 if *either* test statistic rejects its null hypothesis, otherwise accept H_0.
5. If H_0 is not rejected, find the hazard function and, hence, the distribution of the net lifetime of interest by substituting $\hat{p}$ for p and using the maximum likelihood estimates of $f_{Y_T}(t)$,$S_{Y_T}(t)$ and $S_{Y_C}(t)$ in Equation 25.

D'Agostino and Stephens [32] is an excellent reference for goodness-of-fit testing. It contains procedures for many well known distributions and is complete with tables of percentage points and computing formulas as well as recommendations for which tests are generally more powerful for given distributions.

Many models of the net lifetime of interest can be constructed using combinations of popular lifetime distributions such as the Weibull, gamma, lognormal, inverse Gaussian, extreme value, and many others to characterize crude lifetimes. For example, if both of the crude lifetimes are hypothesized to come from independent 2-parameter Weibull distributions with distribution functions $F_{Y_T}(t) = 1 - \mathrm{e}^{-(t/\eta)^{\beta}}, t, \eta, \beta > 0$, and $F_{Y_C}(t) = 1 - \mathrm{e}^{-(t/\alpha)^{\kappa}}, t, \alpha, \kappa > 0$, and the null hypothesis is not rejected, then the hazard function of the net lifetime of interest would have the form

$$h_T(t) = \frac{p\frac{\beta}{\eta^{\beta}}t^{\beta-1}\mathrm{e}^{-(t/\eta)^{\beta}}}{p\mathrm{e}^{-(t/\eta)^{\beta}} + (1-p)\mathrm{e}^{-(t/\alpha)^{\kappa}}}.$$

The relationships between the hazard function and the density, survivor, and distribution functions are well known. Once the hazard function is determined, the density function is given by

$$f_T(t) = h_T(t)\mathrm{e}^{-\int_0^t h_T(\tau)d\tau}$$

and the survivor function of the distribution is

$$S_T(t) = \mathrm{e}^{-\int_0^t h_T(\tau)d\tau}$$

and, of course, the distribution function is $F_T(t) = 1 - S_T(t)$. See Appendix B for details of an interesting phenomenon in which the net lifetime of interest, T, follows a split population model [118]. This phenomenon occurs when the cumulative hazard function of the crude lifetime of the censoring variable is less than the cumulative hazard function of the crude lifetime of the variable of interest. That is, when $H_{Y_C}(t) < H_{Y_T}(t)$.

Although this resulting functional form of the distribution may not be as easy to manipulate as other distributional forms, a wide variety of distribution shapes can be modeled and many types of reliability analyses and maintenance planning can still be performed. Consider the following examples in which the 2-parameter Weibull distribution is used to model the crude lifetimes. Figure 3 shows the density and distribution functions of the net lifetime when the crude failure distribution is Weibull with shape 2.5 and scale 900 and the crude censoring distribution is Weibull with shape 1.5 and scale 600. Figure 4 shows the density and distribution function of the net lifetime when the crude failure distribution is gamma with shape 3 and scale 500 and the crude censoring distribution is lognormal with shape 1.5 and scale 600. Many other distribution shapes can result by varying the Weibull parameters and using different distributions to represent crude lifetimes. The benefits of this approach lie not only in its versatility, but in the fact that the crude lifetimes are simply taken as complete samples and complete sample methods may be used to test the goodness-of-fit of each.

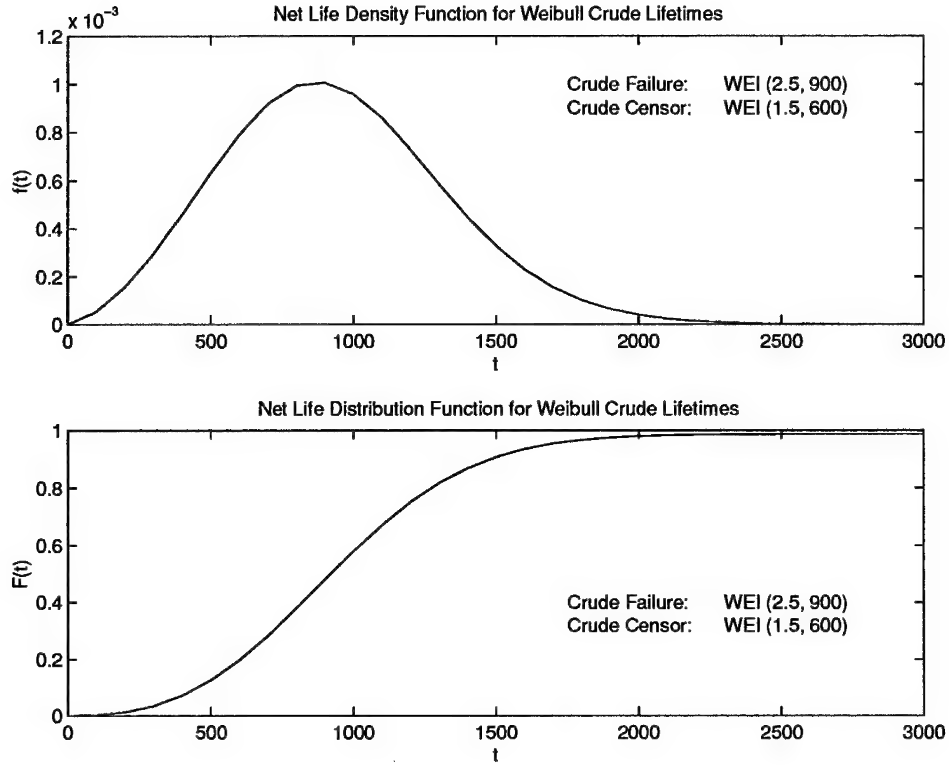

Figure 3 Net Lifetime when Crude Lifetimes are WEI(2.5, 900) and WEI(1.5, 600) at 50% Expected Censoring.

It is important, however, to remember that when conducting a simultaneous test of hypothesis that, as a result of Bonferroni's inequality, an overall level of significance of at most α is maintained by dividing α by the number of tests, two in this case, so the level of significance used in each test should be $\frac{\alpha}{2}$.

An application of this test procedure is demonstrated on a randomly right-censored set of leukemia remission times. The data is taken from [87: p. 190]. In the study, the control group consisted of a sample of 21 patients who were treated with a drug named 6-mercaptopurine (6-MP). The cancer resumed in 9 of the patients while under observation whereas 12 were still in remission until they were no longer observable, or censored. The set of observed remission times (measured in weeks) is

$$6 \quad 6 \quad 6 \quad 7 \quad 10 \quad 13 \quad 16 \quad 22 \quad 23$$

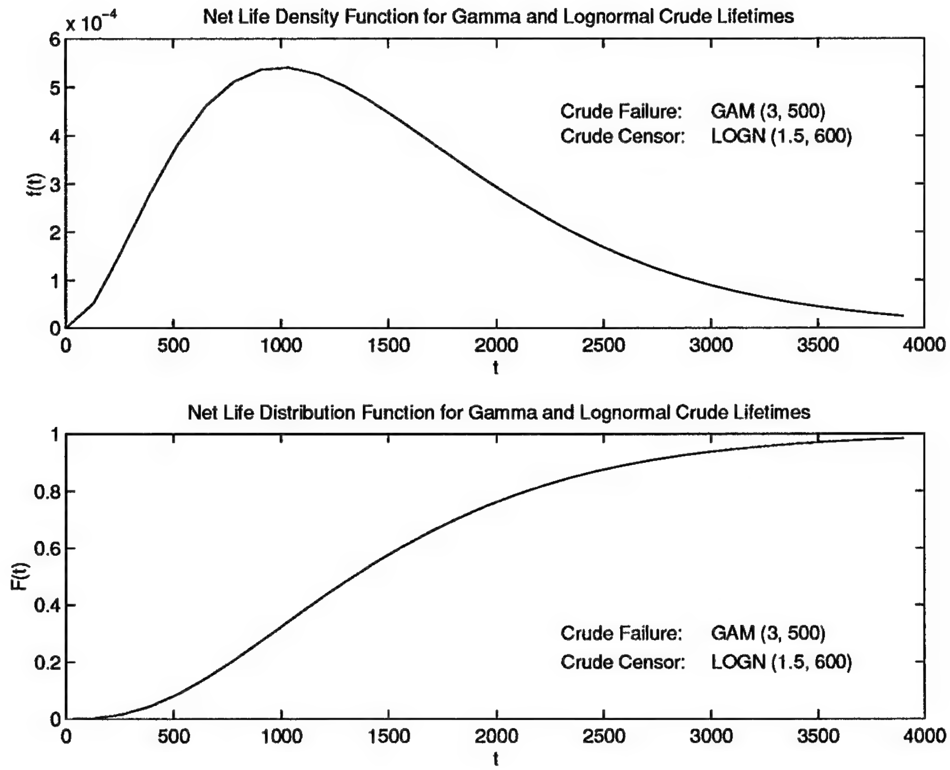

Figure 4 Net Lifetime when Crude Lifetimes are GAM(3, 500) and LOGN(1.5, 600)) at 50% Expected Censoring.

and the set of withdrawal times is

$$6 \quad 9 \quad 10 \quad 11 \quad 17 \quad 19 \quad 20 \quad 25 \quad 32 \quad 34 \quad 35.$$

The set of observed remission times constitutes a complete sample of crude lifetimes for the variable of interest; the remission time of a leukemia patient taking the drug 6-MP. Using the Anderson-Darling test for complete samples, as outlined in [32], a test for the 2-parameter Weibull distribution (transformed to the extreme value distribution) does not reject the composite hypothesis of an underlying Weibull distribution for the crude life of the variable of interest

$$\left(A^2(1+\frac{0.2}{\sqrt{9}}) = 0.5801, 0.10 < p < 0.25\right).$$

Similarly, the Anderson-Darling test with the composite hypothesis of an underlying Weibull distribution on the complete sample of 12 withdrawal times does not reject the Weibull hypothesis $\left(A^2(1+\frac{0.2}{\sqrt{12}}) = 0.4719, p > 0.25\right)$. The maximum likelihood estimates of the parameters of the 2-parameter Weibull distribution for the failure times are $\hat{\beta} = 2.03$ for the shape and $\hat{\eta} = 13.77$ for the scale. Likewise, the estimates of the Weibull parameters for the withdrawal times are $\hat{\kappa} = 2.22$ for the shape and $\hat{\alpha} = 23.60$ for the scale. The expected proportion of censoring is estimated by $\hat{p} = \frac{9}{21}$. Using Equation 25 and numerical integration using Simpson's rule (see Appendix A) ,

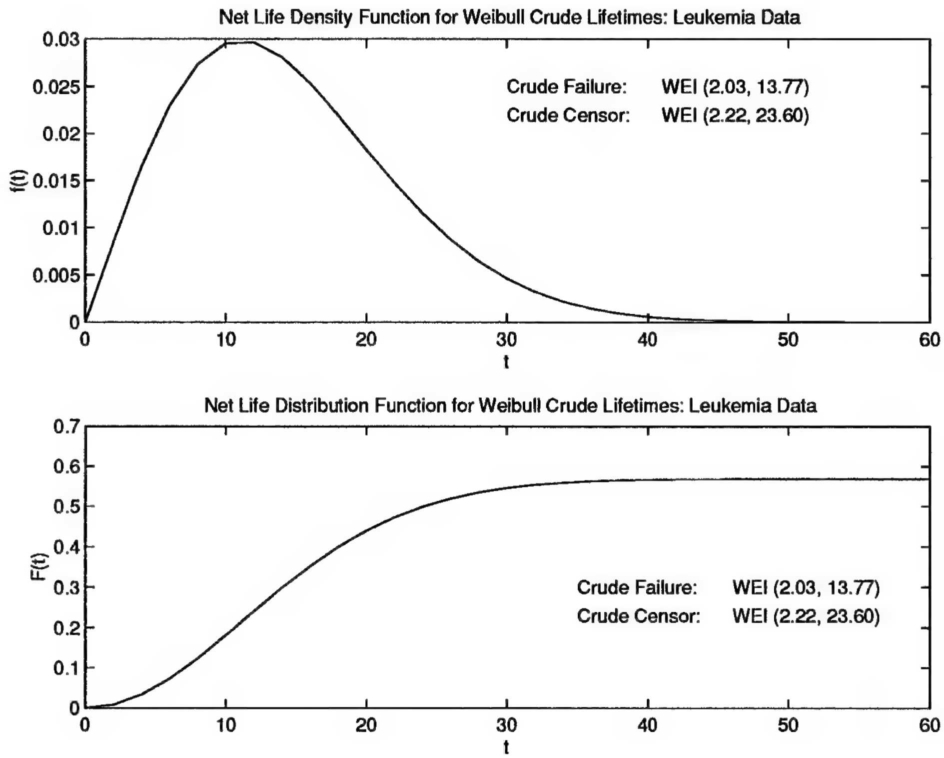

Figure 5 Net Lifetime for the Leukemia Remission Times when Crude Lifetimes are WEI(2.03, 13.77) and WEI(2.22, 23.60)) at 57% Censoring.

plots of the density and distribution functions for the underlying distribution of the net time of remission were constructed and are shown in Figure 5. Further, the distribution function estimate of the underlying net time of remission is plotted with the Kaplan-Meier estimate in Figure 6. The Kaplan-Meier estimate was constructed from the censored set of data with the necessary ad-

justments made to accommodate the ties [87]. The estimate appears quite good in spite of such a small sample and relatively high censoring. Interestingly, the resulting distribution of the net lifetime of remission is a split model, the phenomenon discussed in Appendix B. Split models have been applied in recidivism and economics by Schmidt and Witte [118, 119]. This example indicates that the split model may also be appropriate the leukemia remission times in the study of the drug 6-MP. Statistically speaking, the implication in this example is that the cancer will resume in approximately 58% of the patients who are treated with 6-MP, leaving the remaining 42% leukemia-free for life. Before taking such strong claims too seriously, however, keep in mind the small sample size used in the study.

3.8 New Semi-Parametric Goodness-of-Fit Tests Based on Crude Lifetimes

Other goodness-of-fit tests can be constructed using the crude failure and censoring times where a parametric fit is obtained only for the crude failure times and the EDF of the crude censoring times is used in Equation 25 to find the distribution of the underlying net failure process. This approach eliminates the need to fit a distribution to the crude censoring variable. Furthermore, aside from independence of T, no assumption about the censoring distribution is necessary at all, which distinguishes the following procedure from the rest. The procedure to test the null hypothesis $\mathrm{H}_0 : F_{Y_T} \in \{H_\phi, \phi = (\phi_1, \ldots, \phi_k)' \in \Phi\}$ is simply to perform any complete sample goodness-of-fit test on the set of failure times only and, if H_0 is not rejected, the partially parametric hazard function of the underlying net failure process is then given by

$$h_T(t) = \frac{\hat{p} f_{Y_T}(t, \hat{\phi}_{mle})}{\hat{p} S_{Y_T}(t, \hat{\phi}_{mle}) + (1-\hat{p})\hat{S}_{Y_C,n}(t)} \tag{27}$$

where $\hat{S}_{Y_T,n}(t)$ is the empirical survivor function of the crude lifetime of the censoring variable and is defined as

$$\hat{S}_{Y_T,n}(t) = \begin{cases} 1, & t < c_{(1)} \\ 1 - \frac{i}{n} & c_{(i)} \leq t < c_{(i+1)}, i = 1, \ldots, n-r-1 \\ 0, & t \geq c_{(n-r)}. \end{cases}$$

The power of this test is the same as that of any complete sample test for a given distribution with sample size equal to the size of the failure set in the censored sample.

To demonstrate the usefulness of this procedure, it is applied to a randomly right-censored set of survival time of 211 stage IV prostate cancer patients who have been treated with estrogen in a Veterans Administration Cooperative Urological Research Group study [138]. This data has been examined by several authors. In the study, 90 patients died of prostate cancer, 105 died due to other causes, and 16 were still alive. Therefore, we have 90 failure times and 121 withdrawal times. The actual data set can be found in [42, 66, 138]. Prior analysis in the form of a simple test of hypothesis assuming an underlying exponential distribution with a mean of 100 weeks has been performed by Koziol and Green, Hollander and Proschan, Ebrahimi and Habibullah, and Csörgő and Horváth. Hollander and Proschan performed Hyde's test on the data as well as their own. Further, Hollander and Proschan performed the Koziol and Green test and arrived at different results than Koziol and Green's original calculations [66]. The results of Hollander and Proschan for the Koziol and Green test are the ones given in Table 13. C. H. Chen performed a composite test of hypothesis for the exponential distribution using his correlation test statistic. For comparison purposes, Burke's test for exponentiality was performed under a composite hypothesis as well as the KME-modified Cramér-von Mises and Anderson-Darling tests as presented in this dissertation. The KME-modified Cramér-von Mises and Anderson-Darling tests as well as Burke's, Chen's, and Csörgő & Horváth's tests reject the hypothesis of exponentiality at $\alpha = 0.05$. The results of these tests are summarized in Table 13. Using the set of 90 failure times, measured in weeks, and the Cramér-von Mises test for complete samples, as outlined in [32], a test for the

Table 13 Summary of Test Results for the Prostate Cancer Data.

Test	Hypothesis	p-value	Comments
Koziol & Green	Simple, EXP(100)	0.14	Use $q = 0.60$
Hyde	Simple, EXP(100)	0.86	Use $\hat{q} = 0.573$
Hollander & Proschan	Simple, EXP(100)	0.49	Use $\hat{q} = 0.573$
Ebrahimi & Habibullah	Simple, EXP(100)	0.66	Use $\hat{q} = 0.573$
Csörgő & Horváth	Simple, EXP(100)	0.04	Use $\hat{q} = 0.573$
C. H. Chen	Composite, EXP	0.026	Use $\hat{q} = 0.573$
Burke	Composite, EXP	$p < 0.025$	Use $q = 0.50, n = 100$
$W^2_{r,n}$ (Reineke)	Composite, EXP	$p < 0.025$	Use $q = 0.60, n = 200$
$A^2_{r,n}$ (Reineke)	Composite, EXP	$p < 0.025$	Use $q = 0.60, n = 200$

2-paremeter Weibull distribution (transformed to the extreme value distribution) does not reject the composite hypothesis of an underlying Weibull distribution for the crude life of the prostate cancer survival time $\left(W^2(1 + \frac{0.2}{\sqrt{90}}) = 0.9063, 0.10 < p < 0.25\right)$. The maximum likelihood estimates of the parameters of the 2-parameter Weibull distribution for the prostate cancer survival times are $\hat{\beta} = 0.89$ for the shape and $\hat{\eta} = 34.14$ for the scale. None of the typical lifetime distributions fit the crude life of the censoring times so the semi-parametric crude life procedure was used. The empirical survivor function of the set of censoring times is shown in Figure 7. Figure 8 shows the juxtaposition of distribution function estimates from the semi-parametric crude life procedure, the Kaplan-Meier estimator, the maximum likelihood estimator for the exponential distribution, and the hypothesized exponential distribution with mean 100. It would appear that the estimate from the semi-parametric crude life test more closely resembles the Kaplan-Meier estimate, which should be relatively good for this sample size. This further indicates that the exponential distribution does not appear to be a suitable parametric model for the underlying distribution of the survival time for prostate cancer patients treated with estrogen as in this study.

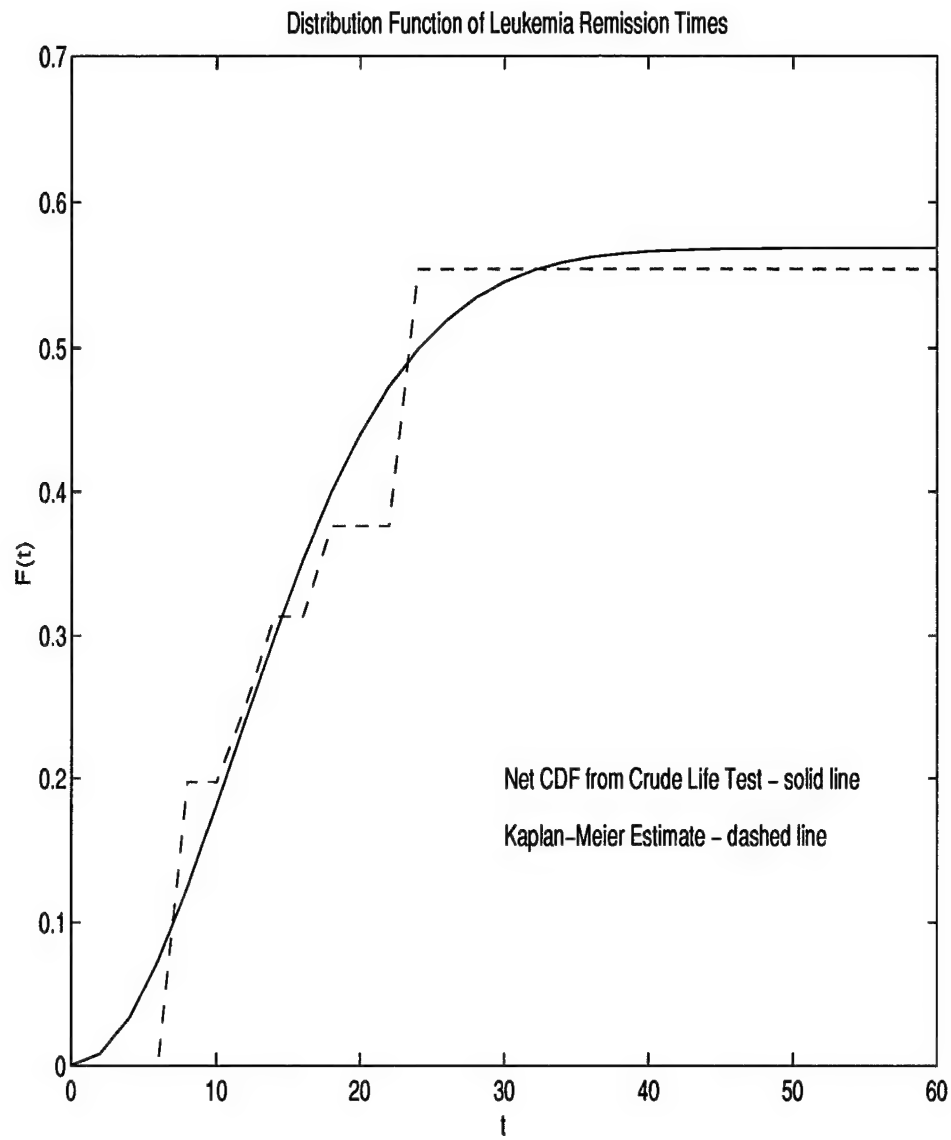

Figure 6 Net Lifetime CDF for Leukemia Remission Times from Crude Lifetimes WEI(2.03, 13.77) and WEI(2.22, 23.60)) at 57% Censoring.

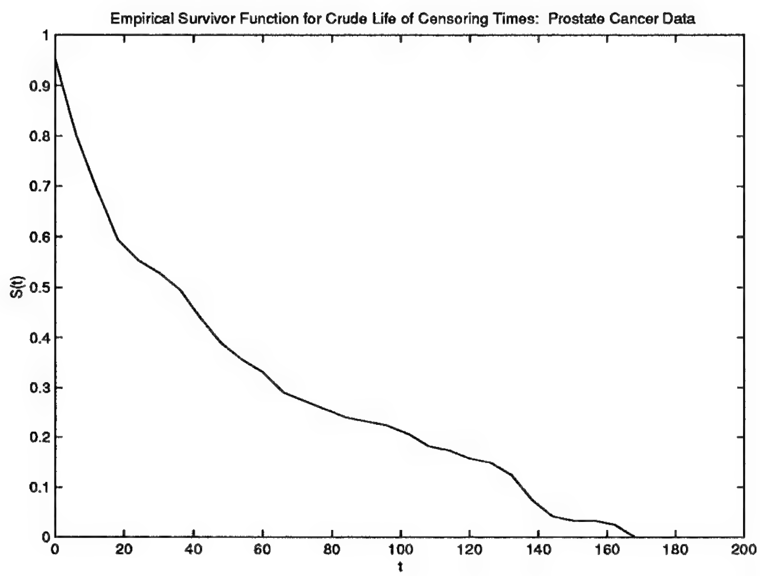

Figure 7 Empirical Survivor function of Crude Life Censoring Times for Semi-Parametric Crude Life Test.

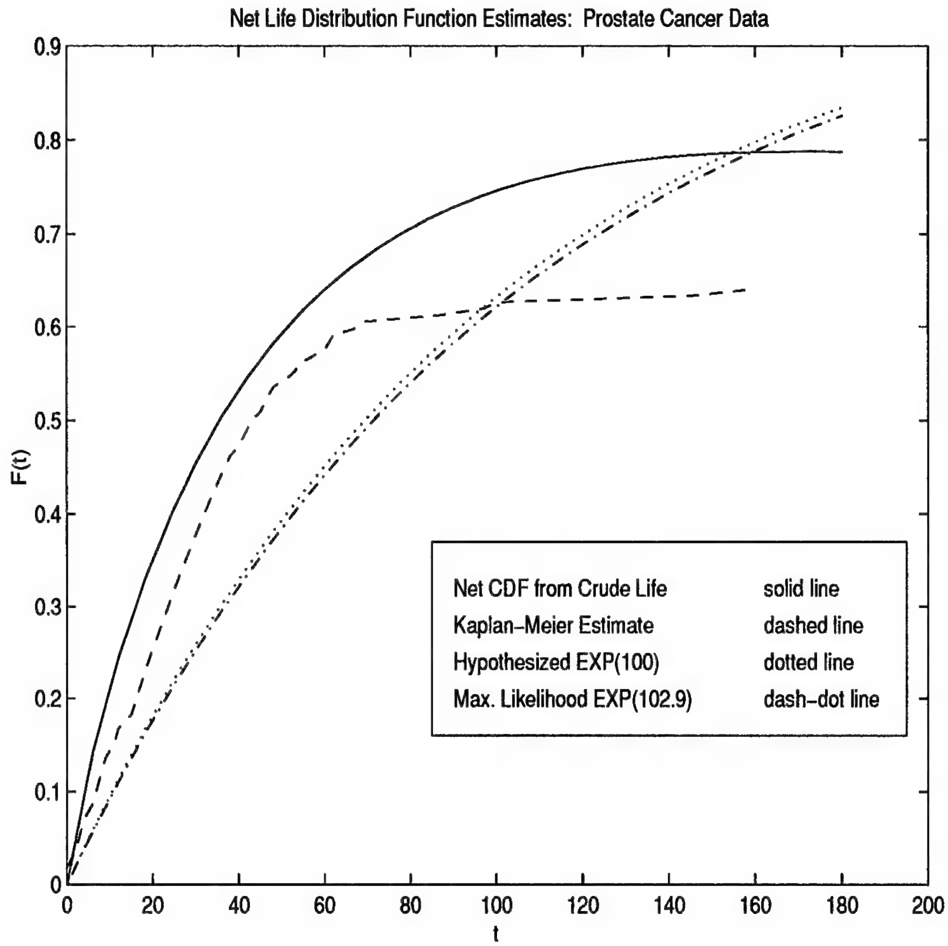

Figure 8 CDF Comparison for Semi-Parametric Crude Life Test with Crude Life WEI(0.89,34.14).

3.9 Power Studies

3.9.1 Exponential Failure with Exponential Censoring. The power of a goodness-of-fit test measures how well the test identifies distributions that are different from the one specified by the null hypothesis. Generally, for one-sided test procedures such as these, we reject the null hypothesis and conclude that the underlying random variable is not exponentially distributed when the observed test statistic is greater than the critical value for a given sample size, proportion of censoring, and significance level. The empirical size of the test is given by the column in which the true distribution is the one specified in the null hypothesis, in this case the exponential distribution. If the test is unbiased, the size should be equal to the level of significance α that was used to determine the percentage point used as the critical value for the hypothesis test.

Goodness-of-fit tests were conducted and compared in a Monte Carlo study to compare tests using the modified Cramér-von Mises and Anderson-Darling statistics to each other as well as to statistics derived by Burke and C.H. Chen and to examine their effectiveness in detecting departures from a randomly censored exponential distribution when the assumption of an exponential censoring distribution is correct. Power studies for the simultaneous tests of crude lifetimes (STCL) of Section 3.7 are also included. An exact comparison of the power of the STCL to the other goodness-of-fit procedures is only possible under the scenario of an exponential lifetime subject to an exponentially distributed random censoring variable. The reason for this is that in the competing risks model with independent risks, exponentially distributed net lifetimes *always* correspond to exponentially distributed crude lifetimes. Thus, we conduct complete sample goodness-of-fit procedures for the exponential distribution with the STCL. As proof of this, consider genaralizing the work of Leemis [87: Example 5.2]. Suppose we have two independent exponentially distributed net lifetimes with survivor functions

$$S_T(t) = \mathrm{e}^{-\frac{t}{\lambda}}, \quad t > 0, \lambda > 0$$

and

$$S_C(c) = \mathrm{e}^{-\frac{c}{\theta}}, \quad c > 0, \theta > 0.$$

The crude survivor function of the risk associated with the net lifetime of interest, T, is

$$\begin{aligned}
S_{Y_T}(y_T) &= \frac{P[X \geq y_T, \delta = 1]}{P[\delta = 1]} \\
&= \frac{P[X \geq y_T, T \leq C]}{P[T \leq C]} \\
&= \frac{\int_{y_T}^{\infty} \left[\int_t^{\infty} \frac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}} \cdot \frac{1}{\theta}\mathrm{e}^{-\frac{c}{\theta}} dc\right] dt}{\int_0^{\infty} \frac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}-\frac{t}{\theta}} dt} \\
&= \frac{\int_{y_T}^{\infty} \frac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}-\frac{t}{\theta}} dt}{\left(\frac{\theta}{\lambda+\theta}\right)} \\
&= \frac{\left(\frac{\theta}{\lambda+\theta}\right)\mathrm{e}^{-\frac{t}{\lambda}-\frac{t}{\theta}}}{\left(\frac{\theta}{\lambda+\theta}\right)} \\
&= \mathrm{e}^{-\frac{y_T}{\lambda}-\frac{y_T}{\theta}}, y_T > 0.
\end{aligned}$$

The crude lifetime associated with the other net risk is found similarly as

$$\begin{aligned}
S_{Y_C}(y_C) &= \frac{P[X \geq y_C, \delta = 0]}{P[\delta = 0]} \\
&= \frac{P[X \geq y_c, C < T]}{P[C < T]} \\
&= \frac{\int_{y_C}^{\infty} \left[\int_c^{\infty} \frac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}} \cdot \frac{1}{\theta}\mathrm{e}^{-\frac{c}{\theta}} dt\right] dc}{\int_0 \infty \frac{1}{\theta}\mathrm{e}^{-\frac{t}{\lambda}-\frac{t}{\theta}} dt} \\
&= \frac{\int_{y_C}^{\infty} \frac{1}{\lambda}\mathrm{e}^{-\frac{c}{\lambda}-\frac{c}{\theta}} dc}{\left(\frac{\theta}{\lambda+\theta}\right)} \\
&= \frac{\left(\frac{\lambda}{\lambda+\theta}\right)\mathrm{e}^{-\frac{t}{\lambda}-\frac{t}{\theta}}}{\left(\frac{\lambda}{\lambda+\theta}\right)} \\
&= \mathrm{e}^{-\frac{y_C}{\lambda}-\frac{y_C}{\theta}}, y_C > 0,
\end{aligned}$$

yielding the same exponential distribution as the crude lifetime for the first risk. Conversely, the only way to obtain exponentially distributed net lifetimes from crude lifetimes using equation 25 is when the crude lifetimes follow *the same* exponential distribution. This is the only way to obtain a constant hazard rate for each net lifetime. With this in mind, power studies can be conducted using the STCL on the exact same distributions and alternatives as the KME-modified Cramér-von Mises and Anderson-Darling tests, Burke's test and Chen's test.

One thousand tests of each type were conducted for sample sizes 20(20)200, expected proportions of censoring 0.10(0.10)0.90, and levels of significance $\alpha = 0.10, 0.05$, and 0.025. The empirical power of each test was observed to be the ratio of the number of times the observed test statistic was bigger than the appropriate percentage point to the total number of tests. Empirical powers are given in Tables 30 through 38 in Appendix F. Figures 43 through 57 in Appendix F display plots of the empirical power of each test for the exponential distribution in each case examined. The censoring distribution in all simulations was exponential with the scale parameter adjusted to provide the amount of expected censoring for each case. It should also be noted that samples with less than two failures were excluded due to computational difficulties. Furthermore, the samples used in the power study were generated using parameter values that matched the *expected* proportion of censoring for each case. The alternative distributions used in the power study for the exponential are plotted with the hypothesized exponential distribution in Figure 9. The Weibull with shape 2 differs the most from the exponential and should therefore be the easiest alternative for the test statistics to detect. The gamma distribution with shape 1.5 and the lognormal distribution are more difficult alternatives to detect, thus the statistics will display lower power against them.

The power of each test is dependent on the sample size, the amount of censoring, and the underlying distribution. Naturally, as the sample size is increased, the power also increases. In contrast, as the level of censoring increases, the power of each test decreases. As with goodness-of-fit tests for complete samples, power is generally higher for alternative distributions whose cumulative

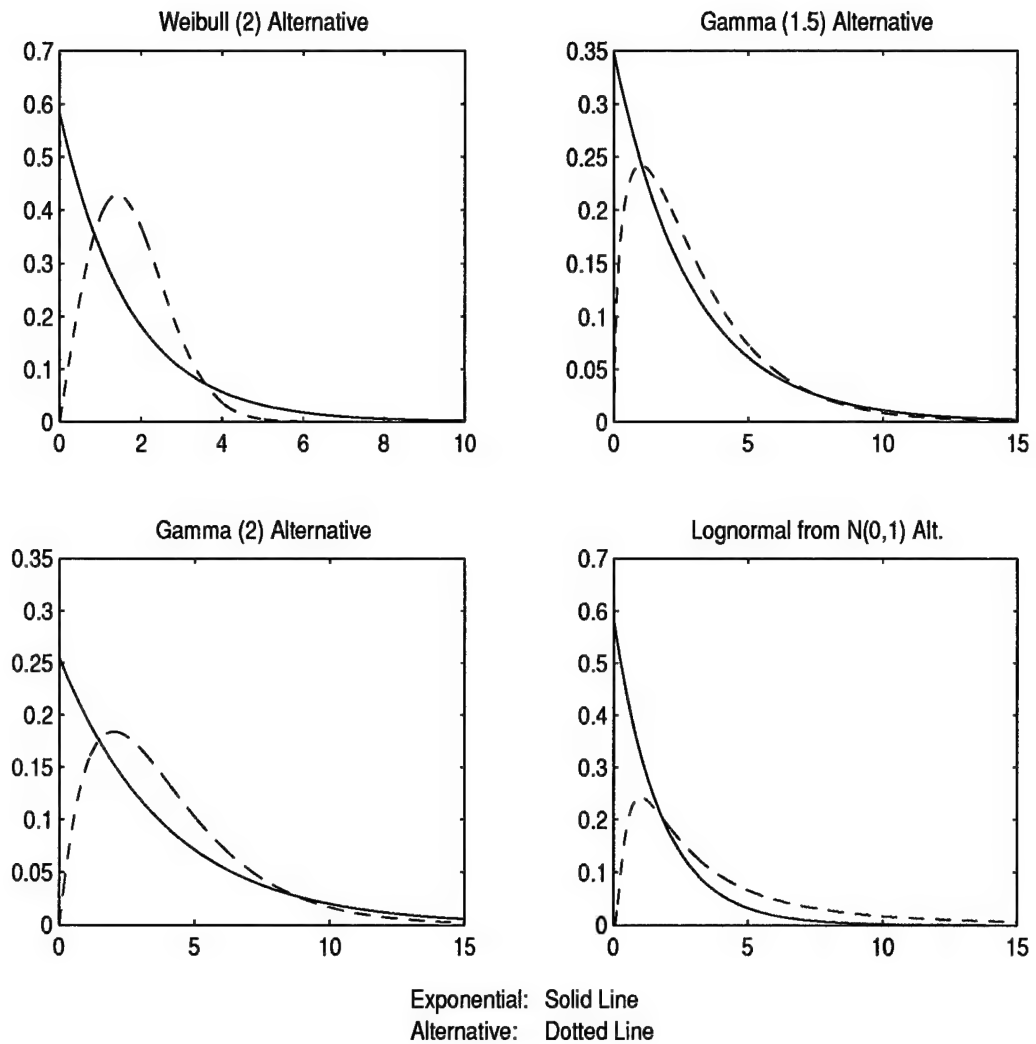

Figure 9 PDF's of Alternative Distributions Used in the Power Study of Tests for Exponentiality.

distribution functions differ to a greater degree in shape than the family specified by the null hypothesis. Keep in mind that when comparing two competing tests for the same set of hypotheses under the same circumstances, the test with the higher power is considered to be better. Therefore, when a relatively powerful goodness-of-fit test exhibits low power against a given alternative distribution, one could interpret that as meaning that the alternative distribution may be just as effective at modeling the underlying process as the distribution specified in the null hypothesis.

3.9.2 Weibull Failure with Exponential Censoring. In the following power study the null hypothesis is that the underlying distribution is from a Weibull family with shape parameter $\beta = 2$. The empirical size of the test is given by the column in which the true distribution is the one specified in the null hypothesis, Weibull with shape 2. If the test is unbiased, the size should be equal to the level of significance α that was used to determine the percentage point used as the critical value for the hypothesis test. The alternative distributions used in the power study for the Weibull with shape 2 are plotted with the hypothesized distribution in Figure 10. All of the alternatives chosen for this power study closely resemble the hypothesised Weibull with shape 2. Empirical powers of $W^2_{r,n}$ and $A^2_{r,n}$observed from 1000 tests of each type are given in tables 14, 15, and 16 for sample sizes 20(20)100 and expected proportions of censoring 0.20, 0.50, and 0.80. The goodness-of-fit tests for the Weibull distribution with unknown location and scale were conducted as outlined in Section 3.6. Unfortunately, distributions of the crude lifetimes that correspond to one Weibull and one exponential net lifetime are not typical parametric distributions for which testing procedures are available, so the STCL and SPCLT procedures are not included in the power studies for the tests for Weibull distributions.

A power study was also conducted for the Weibull family with known shape $\beta = 3.5$. The alternative distributions used in the power study for the Weibull with shape 3.5 are plotted with the more symmetrically shaped version of the Weibull in Figure 11. The normal distribution matches up very closely with the Weibull with shape 3.5 and so we can expect each test statistic to register

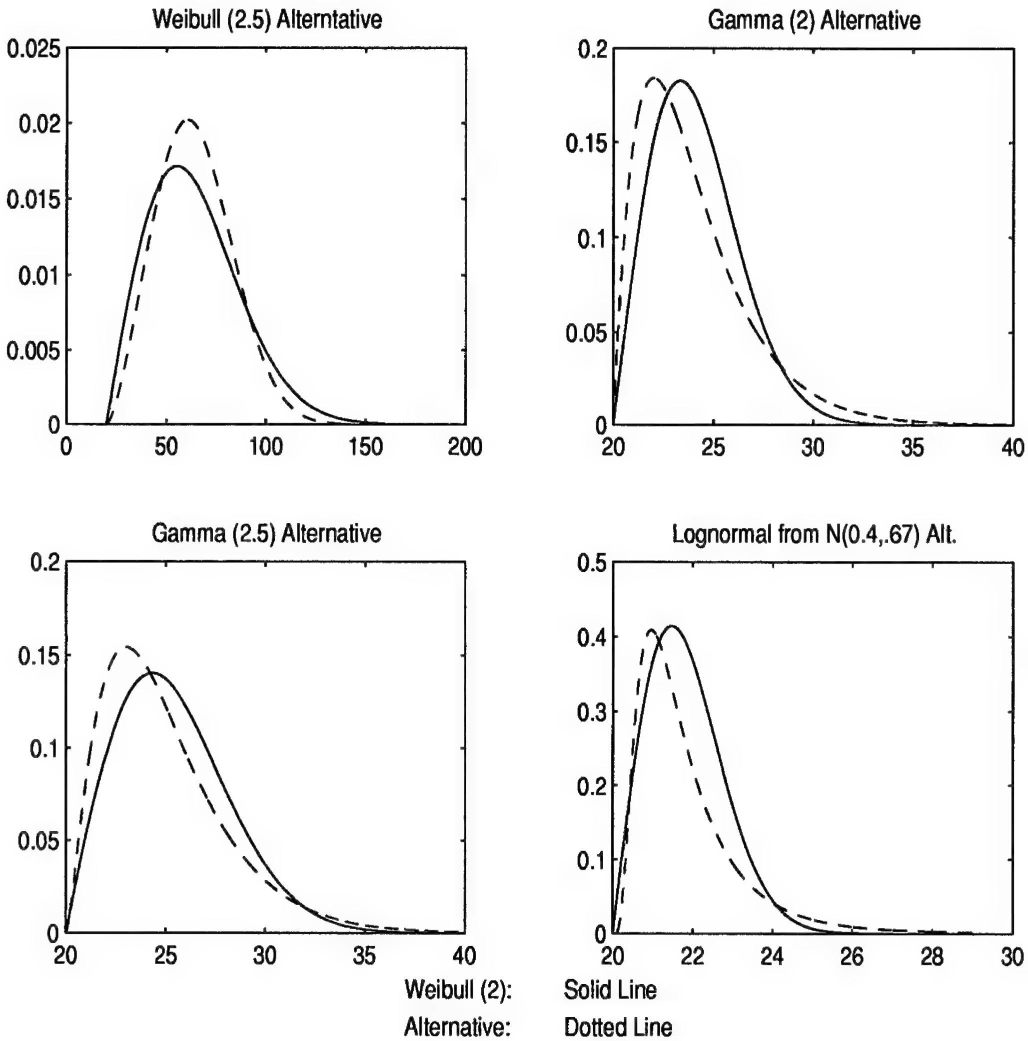

Figure 10 PDF's of Alternative Distributions Used in the Power Study of Tests for the Weibull (shape $\beta = 2$).

Table 14 Empirical Power of Modified $W_{r,n}^2$ and $A_{r,n}^2$ Statistics at $\alpha = 0.10$.
Weibull Distribution (Shape=2) with Exponential Censoring*

	Alternative Distribution									
	Weibull Shape=2		Weibull shape=2.5		Gamma shape=2		Gamma shape=2.5		Lognormal from N(0.4,0.67)	
$q = 0.20$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$
$n = 20$	0.098	0.110	0.111	0.086	0.338	**0.369**	0.225	0.236	0.744	0.735
$n = 40$	0.094	0.090	**0.183**	0.133	0.530	**0.573**	0.334	0.342	0.832	0.813
$n = 60$	0.084	0.075	**0.283**	0.241	0.675	**0.720**	0.417	0.429	0.895	0.884
$n = 80$	0.107	0.100	**0.372**	0.326	0.795	**0.827**	0.519	0.531	0.942	0.941
$n = 100$	0.093	0.101	**0.464**	0.431	0.869	0.896	0.605	0.622	0.968	0.967
$q = 0.50$										
$n = 20$	0.096	0.095	0.090	0.060	0.172	**0.209**	0.130	0.152	0.418	0.423
$n = 40$	0.087	0.092	**0.164**	0.120	0.311	**0.373**	0.180	0.204	0.468	0.480
$n = 60$	0.093	0.090	**0.240**	0.179	0.432	**0.515**	0.262	0.282	0.577	0.579
$n = 80$	0.111	0.108	**0.297**	0.237	0.505	**0.587**	0.288	**0.320**	0.653	0.656
$n = 100$	0.102	0.106	**0.353**	0.309	0.589	**0.677**	0.348	**0.408**	0.711	0.726
$q = 0.80$										
$n = 20$	0.099	0.103	0.072	0.065	0.099	0.105	0.078	0.082	0.073	0.077
$n = 40$	0.102	0.095	0.056	0.047	0.141	0.151	0.106	0.116	0.099	0.105
$n = 60$	0.104	0.099	0.087	0.066	0.164	0.167	0.131	0.134	0.140	0.148
$n = 80$	0.110	0.105	**0.098**	0.064	0.200	0.225	0.129	0.142	0.155	0.152
$n = 100$	0.107	0.099	**0.129**	0.087	0.233	0.248	0.147	0.155	0.164	0.160

*Boldfaced numbers are significantly higher

low power in detecting the normal alternative. However, we should expect relatively high power in the detection of both of the gamma alternatives and moderate power in detecting the change in shape parameter for an underlying Weibull distribution. Empirical powers of $W_{r,n}^2$ and $A_{r,n}^2$ observed from 1000 tests for each sample size, proportion of censoring, and significance level are given in Tables 17, 18, and 19 for sample sizes 20(20)100 and expected proportions of censoring 0.20, 0.50, and 0.80. The goodness-of-fit tests for the Weibull distribution with unknown location and scale were conducted according to the procedure outlined in Section 3.6.

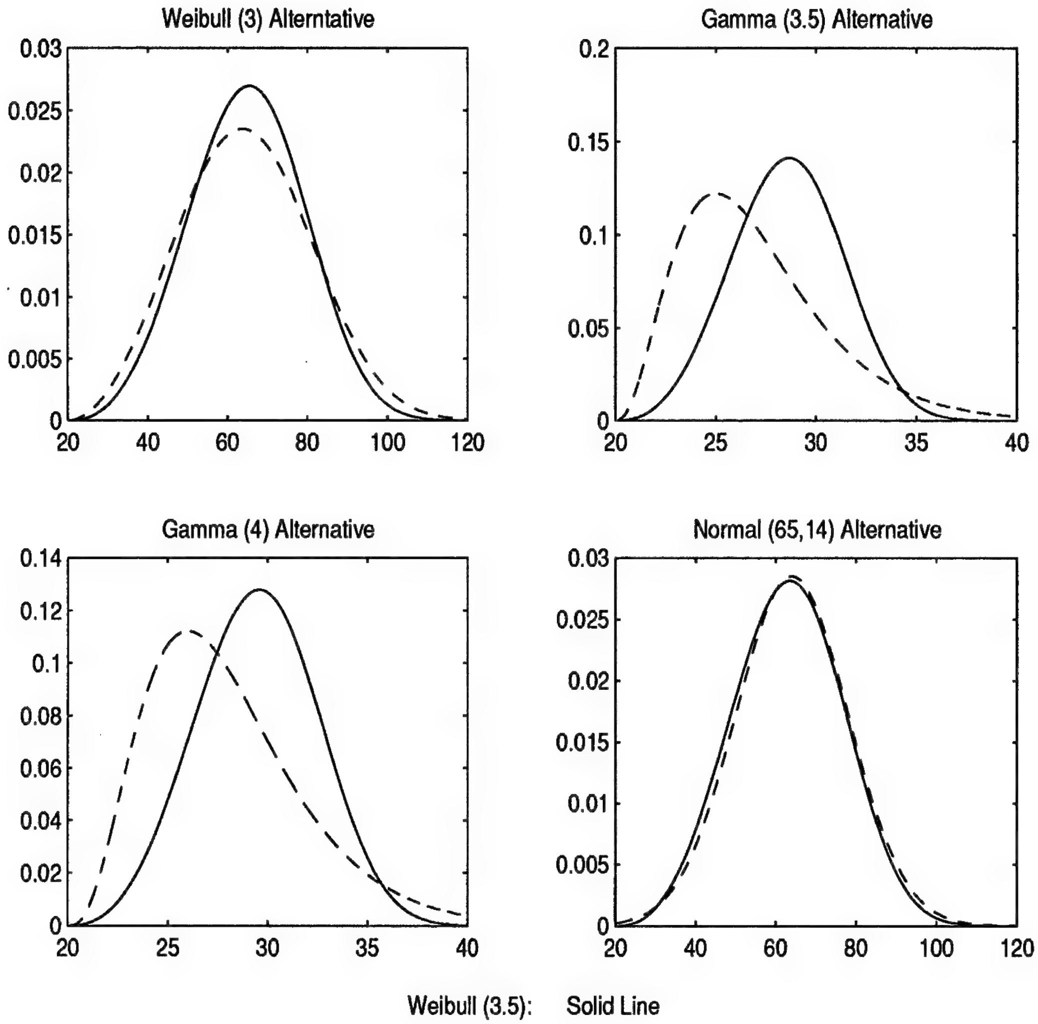

Figure 11 PDF's of Alternative Distributions Used in the Power Study of Tests for the Weibull (shape $\beta = 3.5$).

Table 15 Empirical Power of Modified $W_{r,n}^2$ and $A_{r,n}^2$ Statistics at $\alpha = 0.05$.
Weibull Distribution (Shape=2) with Exponential Censoring*

	Alternative Distribution									
	Weibull Shape=2		Weibull shape=2.5		Gamma shape=2		Gamma shape=2.5		Lognormal from N(0.4,0.67)	
$q = 0.20$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$
$n = 20$	0.056	0.052	0.070	0.044	0.221	**0.255**	0.142	0.148	0.694	0.692
$n = 40$	0.043	0.039	0.108	0.082	0.416	0.429	0.233	0.236	0.779	0.774
$n = 60$	0.036	0.033	0.192	0.167	0.569	**0.606**	0.319	0.320	0.858	0.852
$n = 80$	0.050	0.062	**0.271**	0.238	0.694	**0.731**	0.419	0.421	0.920	0.919
$n = 100$	0.052	0.052	**0.359**	0.326	0.801	0.828	0.492	0.511	0.954	0.948
$q = 0.50$										
$n = 20$	0.056	0.050	0.036	0.019	0.105	0.130	0.066	0.088	0.367	0.359
$n = 40$	0.042	0.048	**0.102**	0.049	0.204	**0.268**	0.100	0.125	0.393	0.403
$n = 60$	0.054	0.048	**0.159**	0.093	0.315	**0.385**	0.165	**0.207**	0.500	0.488
$n = 80$	0.044	0.049	**0.186**	0.139	0.374	**0.450**	0.187	**0.225**	0.570	0.573
$n = 100$	0.043	0.055	**0.234**	0.181	0.483	**0.577**	0.254	**0.298**	0.641	0.632
$q = 0.80$										
$n = 20$	0.054	0.058	0.028	0.023	0.063	0.066	0.033	0.036	0.029	0.030
$n = 40$	0.045	0.047	0.015	0.010	0.079	0.089	0.061	0.063	0.055	0.052
$n = 60$	0.048	0.050	0.026	0.011	0.107	0.116	0.079	0.082	**0.078**	0.063
$n = 80$	0.053	0.055	0.032	0.014	0.126	0.139	0.073	0.089	0.094	0.097
$n = 100$	0.048	0.051	**0.044**	0.013	0.153	0.157	0.085	0.103	0.097	0.104

*Boldfaced numbers are significantly higher

Table 16 Empirical Power of Modified $W_{r,n}^2$ and $A_{r,n}^2$ Statistics at $\alpha = 0.025$.
Weibull Distribution (Shape=2) with Exponential Censoring*

	Alternative Distribution									
	Weibull Shape=2		Weibull shape=2.5		Gamma shape=2		Gamma shape=2.5		Lognormal from N(0.4,0.67)	
$q = 0.20$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$	$W_{r,n}^2$	$A_{r,n}^2$
$n = 20$	0.028	0.030	0.035	0.023	0.160	0.166	0.098	0.111	0.649	0.645
$n = 40$	0.021	0.019	**0.074**	0.042	0.322	0.332	0.180	0.172	0.745	0.733
$n = 60$	0.014	0.015	**0.144**	0.104	0.457	**0.496**	0.238	0.244	0.825	0.821
$n = 80$	0.028	0.034	0.188	0.158	0.591	**0.628**	0.321	0.325	0.894	0.893
$n = 100$	0.027	0.027	0.262	0.240	0.712	**0.751**	0.400	0.413	0.930	0.935
$q = 0.50$										
$n = 20$	0.022	0.023	0.018	0.007	0.063	0.076	0.036	0.053	**0.311**	0.276
$n = 40$	0.026	0.025	0.048	0.021	0.129	**0.171**	0.063	0.085	0.351	0.349
$n = 60$	0.031	0.024	0.097	**0.047**	0.219	**0.290**	0.109	**0.144**	0.431	0.426
$n = 80$	0.025	0.023	0.126	**0.072**	0.277	**0.346**	0.124	**0.159**	0.493	0.510
$n = 100$	0.023	0.028	0.149	**0.093**	0.377	**0.461**	0.175	**0.218**	0.575	0.566
$q = 0.80$										
$n = 20$	0.033	0.033	0.011	0.007	0.030	0.033	0.022	0.022	0.010	0.010
$n = 40$	0.023	0.024	0.006	0.005	0.042	0.054	0.037	0.042	0.024	0.024
$n = 60$	0.021	0.024	0.006	0.002	0.069	0.078	0.040	0.047	0.031	0.024
$n = 80$	0.028	0.027	0.009	0.003	0.067	0.076	0.038	0.038	0.056	0.053
$n = 100$	0.025	0.029	0.012	0.002	0.092	0.100	0.060	0.059	0.067	0.059

*Boldfaced numbers are significantly higher

Table 17 Empirical Power of Modified $W^2_{r,n}$ and $A^2_{r,n}$ Statistics at $\alpha = 0.10$.
Weibull Distribution (Shape=3.5) with Exponential Censoring*

	Alternative Distribution									
	Weibull Shape=3.5		Weibull shape=3		Gamma shape=3.5		Gamma shape=4		N(45,14)	
$q = 0.20$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$
$n = 20$	0.110	0.108	0.185	0.191	**0.519**	0.454	**0.524**	0.468	0.085	0.081
$n = 40$	0.089	0.078	0.279	0.283	**0.786**	0.750	0.822	0.807	0.086	0.067
$n = 60$	0.087	0.095	0.311	0.333	0.880	0.901	0.916	0.921	0.085	0.080
$n = 80$	0.102	0.094	0.420	**0.465**	0.944	0.975	0.959	0.981	0.092	0.081
$n = 100$	0.095	0.107	0.471	**0.519**	0.976	0.999	0.975	0.994	0.099	0.092
$q = 0.50$										
$n = 20$	0.095	0.095	0.154	**0.205**	0.483	**0.559**	0.405	**0.489**	0.101	0.108
$n = 40$	0.093	0.095	0.195	**0.251**	0.736	**0.829**	0.688	**0.790**	0.110	0.090
$n = 60$	0.109	0.101	0.231	**0.298**	0.888	**0.930**	0.851	**0.909**	0.097	0.089
$n = 80$	0.108	0.110	0.293	**0.379**	0.960	0.986	0.928	**0.971**	0.112	0.117
$n = 100$	0.106	0.117	0.325	**0.418**	0.977	0.993	0.965	0.985	0.116	0.117
$q = 0.80$										
$n = 20$	0.115	0.109	0.128	0.153	0.147	**0.181**	0.159	**0.190**	0.107	0.100
$n = 40$	0.104	0.098	0.156	0.184	0.290	**0.342**	0.296	**0.328**	0.105	0.104
$n = 60$	0.088	0.095	0.172	**0.206**	0.401	**0.476**	0.364	**0.442**	0.096	0.096
$n = 80$	0.096	0.099	0.191	**0.245**	0.530	**0.582**	0.461	**0.520**	0.087	0.082
$n = 100$	0.090	0.085	0.217	**0.272**	0.584	**0.653**	0.535	**0.589**	0.092	0.082

*Boldfaced numbers are significantly higher

Table 18 Empirical Power of Modified $W^2_{r,n}$ and $A^2_{r,n}$ Statistics at $\alpha = 0.05$.
Weibull Distribution (Shape=3.5) with Exponential Censoring*

	Alternative Distribution									
	Weibull Shape=3.5		Weibull shape=3		Gamma shape=3.5		Gamma shape=4		N(45,14)	
$q = 0.20$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$
$n = 20$	0.048	0.054	0.104	0.102	**0.395**	0.342	**0.400**	0.328	0.045	0.038
$n = 40$	0.039	0.035	0.161	0.153	**0.673**	0.628	0.716	0.682	0.039	0.033
$n = 60$	0.036	0.041	0.198	0.210	0.784	0.772	0.844	0.840	0.036	0.036
$n = 80$	0.050	0.054	0.271	0.292	0.861	**0.913**	0.903	0.924	0.021	0.020
$n = 100$	0.047	0.055	0.345	**0.398**	0.942	**0.982**	0.940	0.970	0.028	0.024
$q = 0.50$										
$n = 20$	0.048	0.051	0.091	0.119	0.355	**0.431**	0.305	**0.379**	0.057	0.054
$n = 40$	0.049	0.048	0.108	**0.169**	0.644	**0.751**	0.588	**0.684**	0.044	0.044
$n = 60$	0.049	0.045	0.142	**0.207**	0.818	**0.889**	0.774	**0.854**	0.043	0.044
$n = 80$	0.058	0.055	0.164	**0.249**	0.913	**0.959**	0.872	**0.943**	0.059	0.057
$n = 100$	0.057	0.062	0.212	**0.316**	0.956	0.985	0.935	**0.970**	0.063	0.064
$q = 0.80$										
$n = 20$	0.057	0.054	0.083	0.090	0.095	0.123	0.095	0.118	0.054	0.049
$n = 40$	0.055	0.054	0.086	0.112	0.186	**0.235**	0.192	**0.233**	0.055	0.048
$n = 60$	0.046	0.050	0.098	0.125	0.302	**0.364**	0.259	**0.312**	0.044	0.042
$n = 80$	0.047	0.050	0.115	**0.155**	0.411	**0.477**	0.344	**0.406**	0.039	0.036
$n = 100$	0.040	0.040	0.138	**0.181**	0.482	**0.546**	0.425	**0.478**	0.038	0.039

*Boldfaced numbers are significantly higher

Table 19 Empirical Power of Modified $W^2_{r,n}$ and $A^2_{r,n}$ Statistics at $\alpha = 0.025$.
Weibull Distribution (Shape=3.5) with Exponential Censoring*

	Alternative Distribution									
	Weibull Shape=3.5		Weibull shape=3		Gamma shape=3.5		Gamma shape=4		N(45,14)	
$q = 0.20$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$	$W^2_{r,n}$	$A^2_{r,n}$
$n = 20$	0.022	0.034	0.053	0.056	**0.304**	0.229	**0.309**	0.233	0.020	0.017
$n = 40$	0.019	0.015	0.083	0.069	**0.574**	0.520	**0.617**	0.547	0.021	0.017
$n = 60$	0.015	0.020	0.113	0.122	0.689	0.661	0.781	0.763	0.015	0.012
$n = 80$	0.023	0.033	0.196	0.193	0.757	0.786	0.831	0.841	0.021	0.020
$n = 100$	0.025	0.026	0.239	**0.287**	0.881	**0.929**	0.894	0.922	0.028	0.024
$q = 0.50$										
$n = 20$	0.021	0.019	0.048	0.069	0.248	**0.317**	0.232	**0.286**	0.030	0.022
$n = 40$	0.025	0.029	0.062	**0.102**	0.553	**0.657**	0.485	**0.578**	0.023	0.019
$n = 60$	0.028	0.017	0.092	**0.142**	0.750	**0.824**	0.698	**0.796**	0.024	0.019
$n = 80$	0.029	0.026	0.096	**0.164**	0.860	**0.923**	0.826	**0.896**	0.032	0.032
$n = 100$	0.026	0.029	0.141	**0.229**	0.924	**0.963**	0.903	**0.951**	0.035	0.037
$q = 0.80$										
$n = 20$	0.028	0.028	0.048	0.051	0.056	0.076	0.049	0.065	0.020	0.021
$n = 40$	0.024	0.025	0.054	0.060	0.104	**0.151**	0.115	**0.158**	0.026	0.022
$n = 60$	0.025	0.025	0.050	0.071	0.203	**0.267**	0.186	**0.231**	0.016	0.018
$n = 80$	0.023	0.016	0.064	0.085	0.315	**0.376**	0.244	**0.312**	0.014	0.015
$n = 100$	0.017	0.022	0.079	**0.110**	0.379	**0.426**	0.314	**0.386**	0.017	0.015

*Boldfaced numbers are significantly higher

IV. Summary and Conclusions

In addition to a historical perspective on the random censoring problem, maximum likelihood and minimum distance estimation methods for a 3-parameter Weibull distribution subject to an exponentially censoring distribution were presented. Computing formulas were derived for the Cramér-von Mises and Anderson-Darling statistics to be used as distance estimators in the case of randomly censored samples with the EDF in each statistic replaced by the Kaplan-Meier estimator. It was demonstrated that minimum distance estimation can be effectively used to estimate location parameters when samples contain censored items, particularly when used in tandem with maximum likelihood estimation of the shape and scale parameters. The effectiveness of that technique was demonstrated for sample sizes 20 and 60 with expected proportions of censoring 0.25, 0.50, and 0.75. A comparison of initial location parameter estimates for the minimum distance estimation procedure showed that the technique is robust to the choice of initial estimates of the location parameter. The comparison also revealed that there is no significant difference in the estimators regardless of whether the Cramér-von Mises or the Anderson-Darling distance measure is used.

Four known nonparametric distribution function estimators were outlined in addition to the introduction of a new continuous estimator in the form of a trigonometrically-smoothed and jack-knifed product-limit estimator. Two semi-parametric estimators were presented and all of the estimation techniques were compared under various levels of censoring and for a variety of distribution shapes using mean integrated squared error, Kruskal-Wallis tests, and side-by-side boxplots to determine differences in ISE among the estimators. Due to the inherent skewness of ISE, we do not recommend comparing distribution function estimators on the basis if MISE alone. We recommend that Kruskal-Wallis tests be used to compare ISE populations to judge the relative effectiveness estimators in addition to the examination of side-by-side boxplots. The results of that study show that maximum likelihood is the best estimator given that the distribution is correctly specified. The comparison of estimators also revealed that the PEXE and FRWE are the best nonparamet-

ric estimators for both skewed and symmetric underlying distributions under light, moderate, and heavy censoring conditions. However, the FRWE and BSE require considerable computer time to construct in comparison with the non-kernel-type estimators. Trigonometric smoothing was shown to neither improve nor diminish the ability of the Kaplan-Meier estimator in terms of ISE while the jackknifing of the trigonometrically-smoothed KME increased the ISE at 25% expected censoring. The quality of distribution function estimation produced by the MONE is only worse than the KME when censoring is high, otherwise the two methods are not significantly different in terms of ISE. The kernel estimator of Blum and Susarla was the worst of the estimators, especially at 50% and 80% expected censoring. The KME is the easiest and fastest to compute while the kernel-type estimators require the most computer time to construct.

A discussion of the asymptotic theory of goodness-of-fit statistics for composite tests of hypothesis based on the difference between the Kaplan-Meier estimator and the maximum likelihood estimator was provided. Current theory was summarized for the exponential distribution when the censoring distribution is also exponential and for the Weibull distribution when the censoring distribution has a hazard function that is proportional that of the failure distribution. The Efron transform, which was adapted from Doob's work in stochastic processes, is a method of transforming a general Gaussian process to a Brownian motion process. That is, the limiting behavior of the Kaplan-Meier estimator can be transformed to a known process for which percentage points can be obtained analytically in the case of a simple hypothesis. Unfortunately, it is shown that this transform is no longer possible when the hypothesis is composite and parameters are estimated. Problems with the Efron transform in the case of estimated parameters were discussed for both the Weibull and exponential distributions.

Finally, several new goodness-of-fit tests were developed and introduced. This class of new tests is based on replacing the EDF with the KME in the Cramér-von Mises and Anderson-Darling statistics since the EDF is no longer available when samples are randomly censored. Although the

PEXE and FRWE performed significantly better than the KME in the comparison of estimators, the KME was chosen as the substitute for the EDF in these goodness-of-fit statistics because it is very easy to compute, numerical methods are not required, exact computing formulas are available, it reduces to the EDF when no censoring is present, and it is known to converge more rapidly than the FRWE [48]. Procedures for testing the hypothesis of exponentiality are given in the case of a exponentially distributed censoring distribution. Procedures are also given for testing the null hypothesis of a Weibull distribution with shape 2 and a Weibull with shape 3.5, both in the case of an exponentially distributed random censoring variable. As part of these new tests, computing formulas are derived and the location and scale invariance of the new KME-modified Cramér-von Mises and Anderson-Darling statistics are examined. Percentage points are obtained through Monte Carlo simulation for tests for the exponential with estimated scale parameter for sample sizes 20(20)200 and proportions of censoring 0.10(0.10)0.90. Likewise, percentage points are obtained for tests for the Weibull distribution with shape 2 and estimated location and scale and for the Weibull distribution with shape 3.5 with estimated location and scale parameters for sample sizes 20(20)100 and proportions of censoring 0.10(0.10)0.90. This class of goodness-of-fit tests requires the assumption of an exponentially distributed random censoring variable. Some justifications of this assumption are offered through Drenick's Theorem [37] in Section 3.3.

For the test of exponentiality with an exponential censoring distribution, the powers of the KME-modified Cramér-von Mises and Anderson-Darling statistics were compared with the powers of existing procedures by Burke and C. H. Chen. The simultaneous test of crude lifetimes (STCL) was also performed on the exponential distribution with an exponential censoring variable using both the complete sample Cramér-von Mises and Anderson-Darling test statistics. Chen's test had the lowest power but a comparison was only available at 20% censoring. Burke's test, too, had lower power than the KME-modified statistics in most cases, particularly for the sample sizes under 60. Burke's test did, however, tend to catch up in terms of power for samples of size 80 or more. The KME-modified Cramér-von Mises and Anderson-Darling statistics are nearly equivalent in power

and displayed relatively higher power, in general, than the statistics proposed by Burke and Chen. The empirical powers of the STCL tests were either statistically equal to that of the other tests or statistically greater than all of the other tests, but were never significantly lower than any other goodness-of-fit test in the study.

When testing the hypothesis of an underlying Weibull distribution under the assumption of an exponentially distributed censoring variable, there were no existing tests to compare the KME-modified Cramér-von Mises and Anderson-Darling statistics to. For an underlying Weibull distribution with shape 2, the KME-modified Cramér-von Mises dominated the KME-modified Anderson-Darling in detecting a Weibull with shape 2.5. Conversely, the KME-modified Anderson-Darling was dominant over the KME-modified Cramér-von Mises in detecting a gamma distribution with shape 2 when testing for the Weibull with shape 2. It is evident in Tables 17, 18, and 19 that the KME-modified Anderson-Darling performed significantly better in terms of power than the KME-modified Cramér-von Mises in many cases.

Another class of goodness-of-fit tests was constructed using the competing risks concept of crude lifetimes for the failure and censoring variables. Simultaneous tests of fit are performed on the crude lifetimes using the failure set and the withdrawal set each taken as a complete sample. The censoring distribution is no longer assumed to be exponentially distributed but it is still necessary to some parametric family to the crude lifetimes of the censored items. After fitting distributions to both the crude failure and censoring times, one can obtain the hazard function, and hence the density, survivor and distribution functions, of the underlying variable of interest. This is referred to in competing risks theory as a net lifetime and represents the distribution of the variable of interest if it was the only risk acting on the population. That is, if it was uncensored. The benefit of this type of test is that existing complete sample goodness-of-fit procedures may be used and tables of percentage points are readily available. Another advantage is that the power of goodness-of-fit tests of this type is not dependent on the amount of censoring or the assumption of an

exponentially distributed censoring variable. This procedure was applied to the case of testing for an underlying exponential distribution censored by an exponentially distributed random variable and displayed power at least as high as the KME-modified Cramér-von Mises and Anderson-Darling tests. The drawback is that numerical methods will most likely be needed to convert the crude lifetime distributions to the net lifetime distribution of the underlying process and the resulting density, survivor, and distribution functions may not be very easy to manipulate. Nevertheless, many types of reliability analyses and maintenance planning may still be performed within this framework.

Finally new goodness-of-fit procedures are developed again based on the relationship of the crude lifetimes to the net lifetimes. These tests are considered partially parametric in that they require no parametric assumption on the distribution of the censoring random variable. The empirical survivor function of the crude withdrawal times is all that is necessary to adjust the crude lifetime distribution to yield the net lifetime distribution of the underlying cause of failure. The hypothesis is conducted on the crude failure times only. Existing complete sample goodness-of-fit procedures may be used for this and the power of this type of test is the same as that of any test on a complete sample of the same sample size.

Using crude lifetimes to estimate and characterize the net lifetime of interest in cases of randomly censored data broadens the field of reliability analysis in such a way as to include heavily censored data that would have previously been unusable. It also brings much greater flexibility to goodness-of-fit testing with randomly censored samples because any of the existing goodness-of-fit tests for complete samples may be used to try to characterize the crude lifetimes, which are then used to characterize the net lifetime of interest. Our recommendation when faced with performing reliability analysis on a set of randomly censored data is to attempt the simultaneous test of crude lifetimes first. If no parametric model will adequately characterize the set of withdrawals, then

the semi-parametric crude life procedure should be used to estimate the survivor function of the variable of interest.

Appendix A. Numerical Integration with Simpson's Rule

Simpson's Rule is a method for evaluating a definite integral $\int_a^b f(x)dx$ using a parabolic approximation. Simpson's Rule is defined as [9]

$$\int_a^b f(x)dx = [f(x_1) + 4f(x_2) + 2f(x_3) + \ldots + 4f(x_n) + f(x_{n+1})]\frac{\Delta x}{3}$$

where $x_1, \ldots, x_{n+1}$ are values with equal spacing $\Delta x = \frac{(b-a)}{n}$ such that $x_1 = a$, $x_{n+1} = b$, and n is an even integer. Note that the function evaluations are weighted such that the first and last weights are 1 while the weights for the middle terms are alternating 4's and 2's beginning and ending with 4.

Appendix B. An Interesting Net Lifetime Result from Tests Based on Crude Lifetimes

An interesting phenomenon results when the cumulative hazard function of the crude lifetime of the censoring variable is less than the cumulative hazard function of the crude lifetime of the variable of interest, namely, when $H_{Y_C}(t) < H_{Y_T}(t)$. The resulting net lifetime is known as a split population model, or just split model, and has been studied by Schmidt and Witte [118,119]. The analytical foundations of this phenomenon surface in the following way. The hazard function for the underlying net lifetime of the random variable of interest, T, is determined through the relationship

$$h_T(t) = \frac{pf_{Y_T}(t)}{pS_{Y_T}(t) + (1-p)S_{Y_C}(t)} \tag{28}$$

where Y_T is the crude lifetime of the variable of interest and Y_C is the crude lifetime of the censoring random variable. Equation 28 can be expressed as

$$h_T(t) = \frac{ph_{Y_T}(t)S_{Y_T}(t)}{pS_{Y_T}(t) + (1-p)S_{Y_C}(t)}$$

which, in turn, can be written as

$$h_T(t) = \frac{ph_{Y_T}(t)\mathrm{e}^{-H_{Y_T}(t)}}{p\mathrm{e}^{-H_{Y_T}(t)} + (1-p)\mathrm{e}^{-H_{Y_C}(t)}}$$

and finally

$$h_T(t) = \frac{h_{Y_T}(t)}{1 + \left(\frac{1-p}{p}\right)\mathrm{e}^{-H_{Y_C}(t)+H_{Y_T}(t)}}. \tag{29}$$

Now, as $t \to \infty$, the exponential term in the denominator will dominate the expression and if $H_{Y_C}(t) < H_{Y_T}(t)$, then $\lim_{t\to\infty} h_T(t) < \infty$. Consequently, $\lim_{t\to\infty} F_T(t) < 1$. The dominance of the exponential term in the denominator is affected by the proportion of observed failures in the sample, p, as well. As p gets closer to 1, $\lim_{t\to\infty} F_T(t)$ will also get closer to 1. An example of a

net lifetime cumulative distribution function is shown in Figure 12 to illustrate. In the example, the crude censoring variable follows a Weibull distribution with shape 2.5 and scale 25 while the distribution of the crude lifetime of the variable of interest is Weibull with shape 2 and scale 15. Figure 12 makes it clear that only about 68% of the population would fail from the underlying risk of interest if it were the only risk acting on the population (while the remaining members of the population live forever). Although this does not meet the criterion of what we usually consider to be a "legitimate" lifetime distribution, a plausable explanation for this phenomenon may be that a certain proportion of the population will not fail as a result of the underlying variable of interest when that is the only risk present. Consider a medical follow-up study in which there is a randomly censored sample consisting of observed times of death and withdrawal times. It is possible that if we were able to observe all of the original subjects in the sample without any censoring that a certain proportion of them would *never die of cancer.* In essence, for all practical purposes the resulting distributon and survivor functions yield useful results and should be used in reliability and survival analysis.

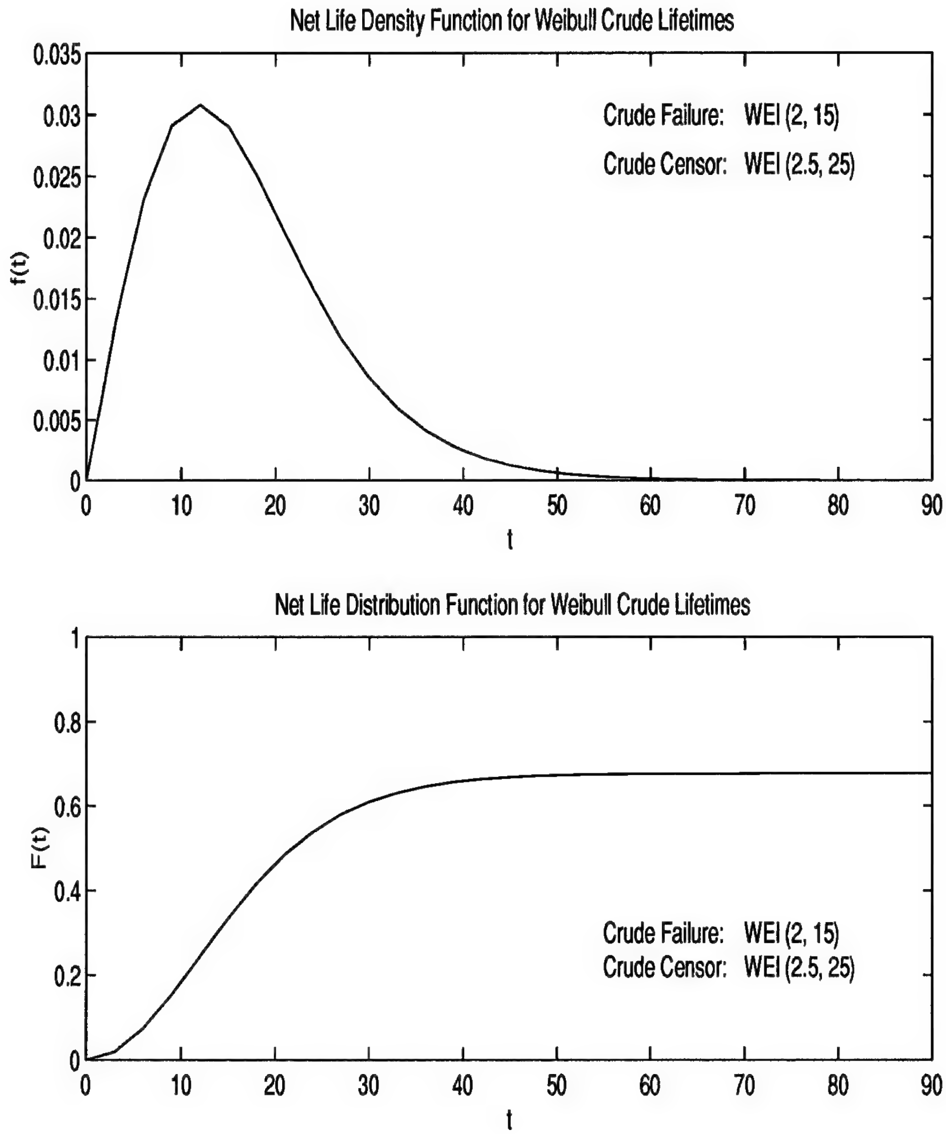

Figure 12 Net Lifetime when the Crude Censoring Lifetime is WEI(2.5, 25) and the Crude Lifetime of Interest is WEI(2, 15) at 50% Expected Censoring.

Appendix C. Plots Illustrating Estimation Techniques for Randomly Censored Data

Plots illustrating examples of all of the distribution function techniques, with the exception of the 3-parameter Weibull, are presented in this Appendix. Each estimator was used to find a distribution funtion estimate for the exponential, Weibull (with shape 2), and Weibull (with shape 3.5) disributions. With each underlying distribution, an exponentially distributed random censoring variable was used and the scale parameter was adjusted to provide the desired expected proportion of censoring. The plots demonstrate the effectiveness of each estimator when the sample is expected to be 25%, 50%, and 75% censored. Samples of size 40 were used in each case.

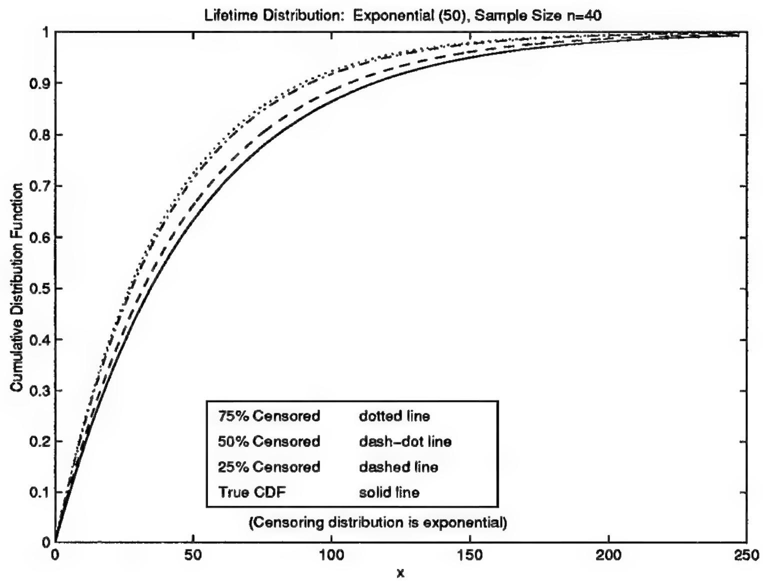

Figure 13 Maximum Likelihood Estimator, Exponential Distribution.

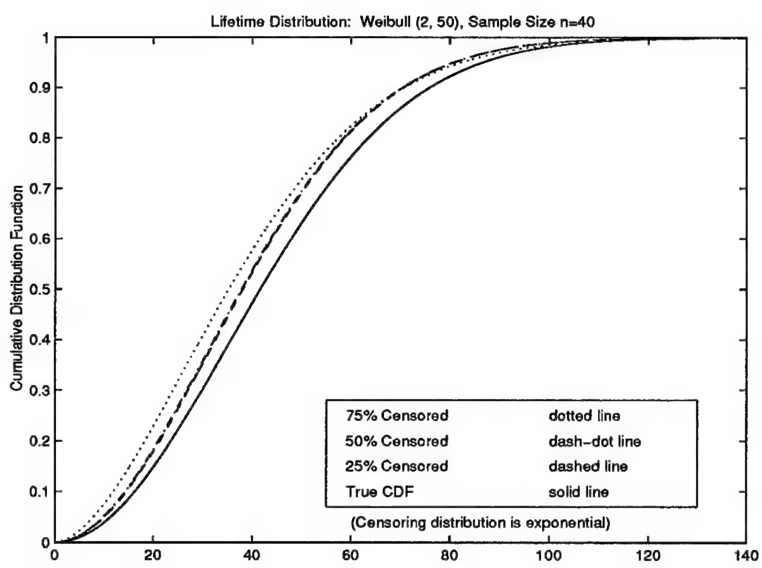

Figure 14 Maximum Likelihood Estimator, Weibull with shape 2.

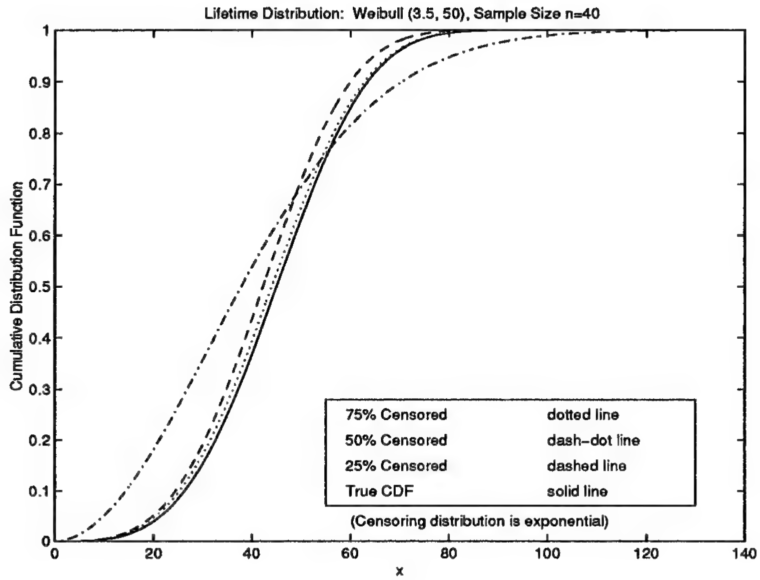

Figure 15 Maximum Likelihood Estimator, Weibull with shape 3.5.

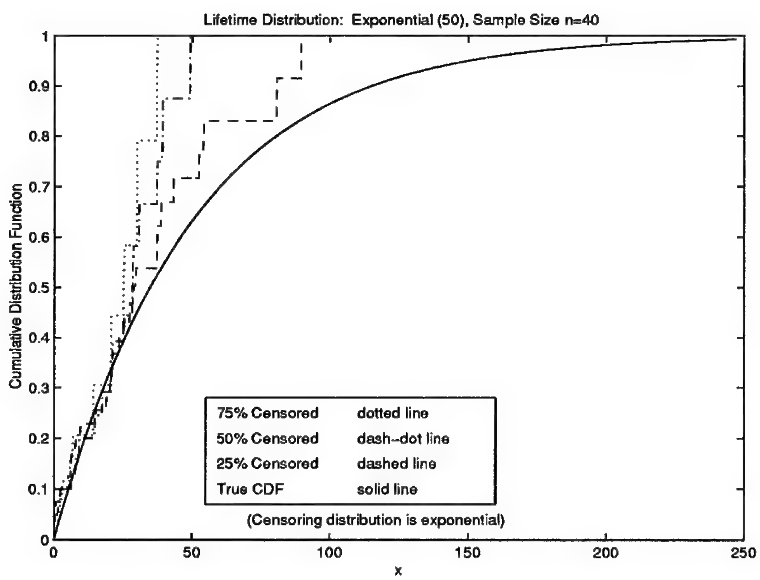

Figure 16 Kaplan-Meier Estimator, Exponential Distribution.

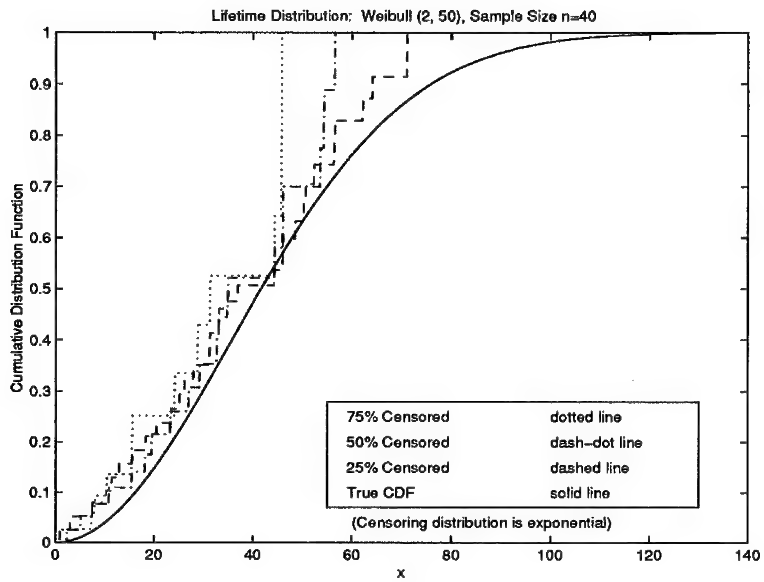

Figure 17 Kaplan-Meier Estimator, Weibull with shape 2.

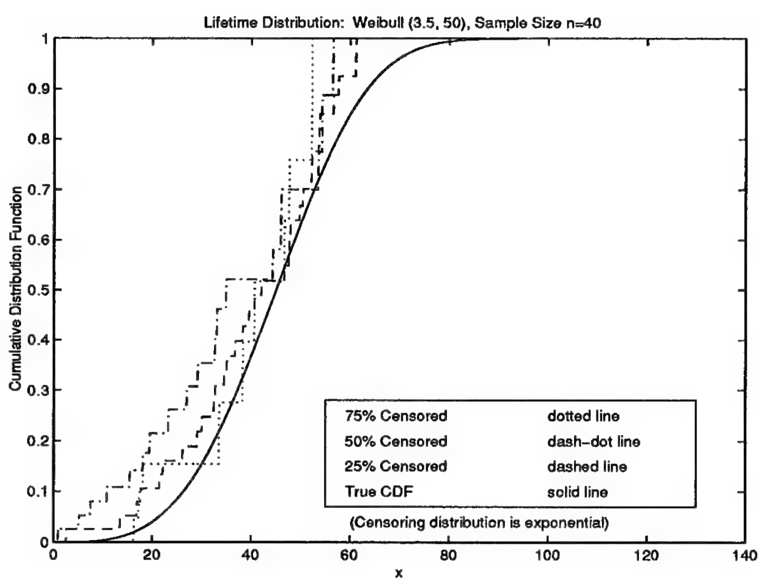

Figure 18 Kaplan-Meier Estimator, Weibull with shape 3.5.

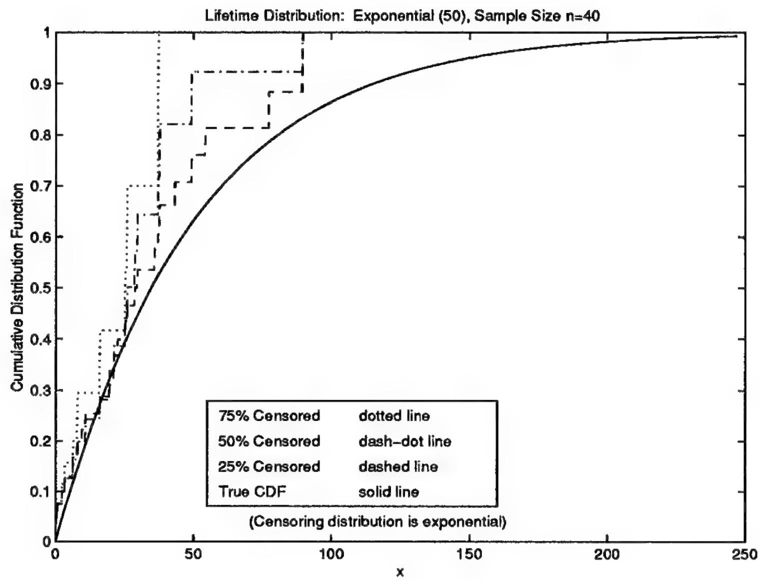

Figure 19 Mean Order Number Estimator, Exponential Distribution.

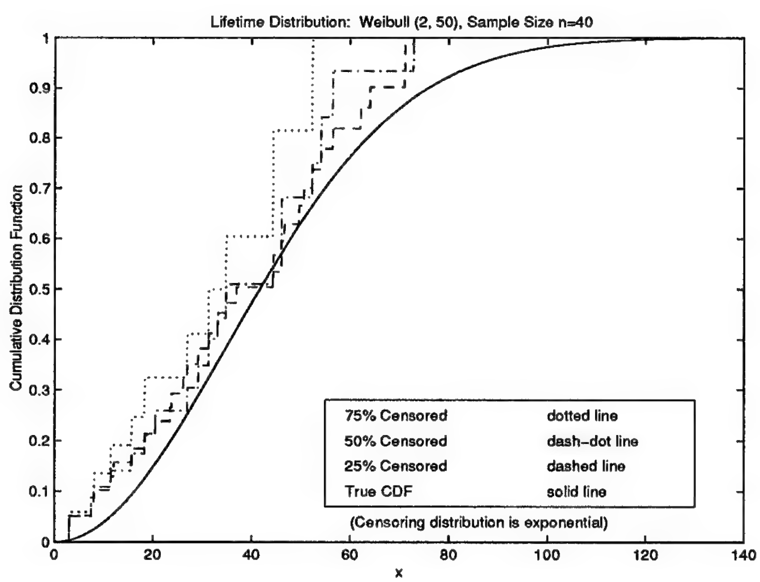

Figure 20 Mean Order Number Estimator, Weibull with shape 2.

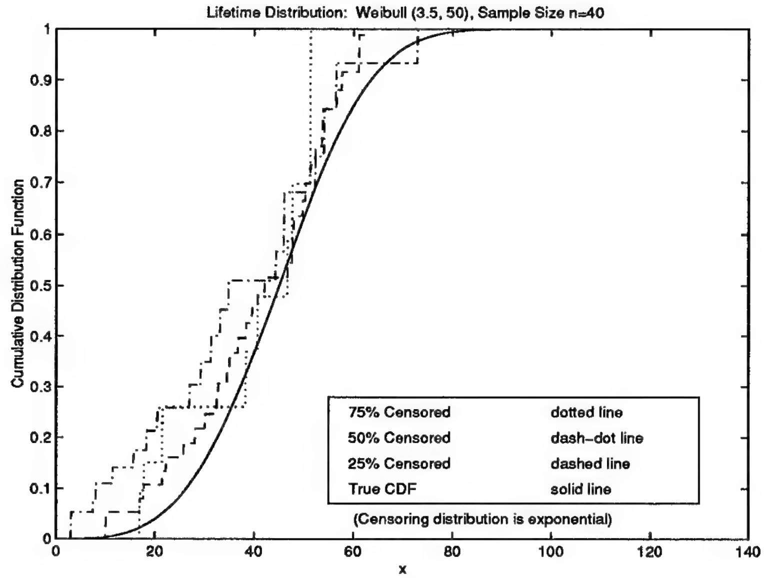

Figure 21 Mean Order Number Estimator, Weibull with shape 3.5.

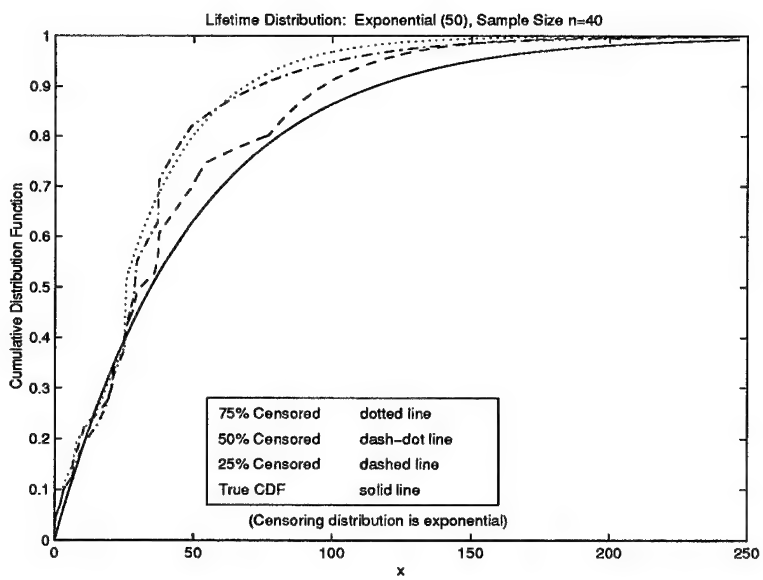

Figure 22 Piecewise Exponential Estimator, Exponential Distribution.

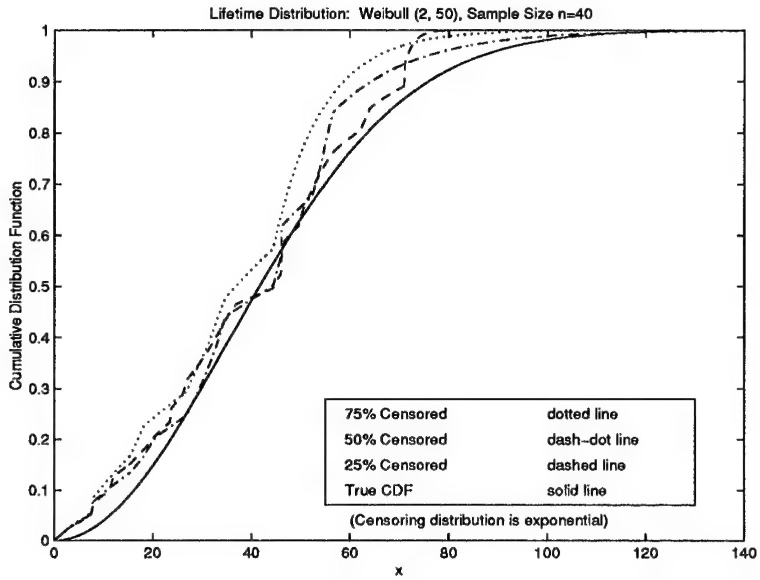

Figure 23 Piecewise Exponential Estimator, Weibull with shape 2.

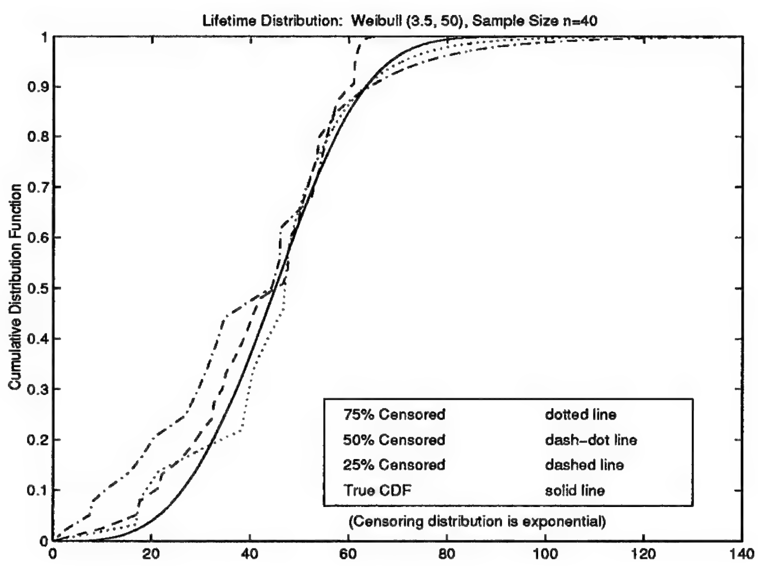

Figure 24 Piecewise Exponential Estimator, Weibull with shape 3.5.

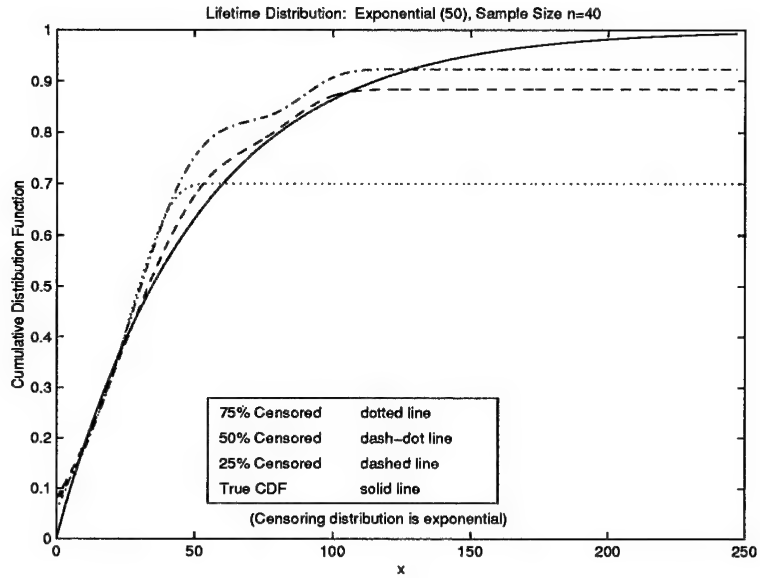

Figure 25 Blum-Susarla Kernel Estimator, Exponential Distribution.

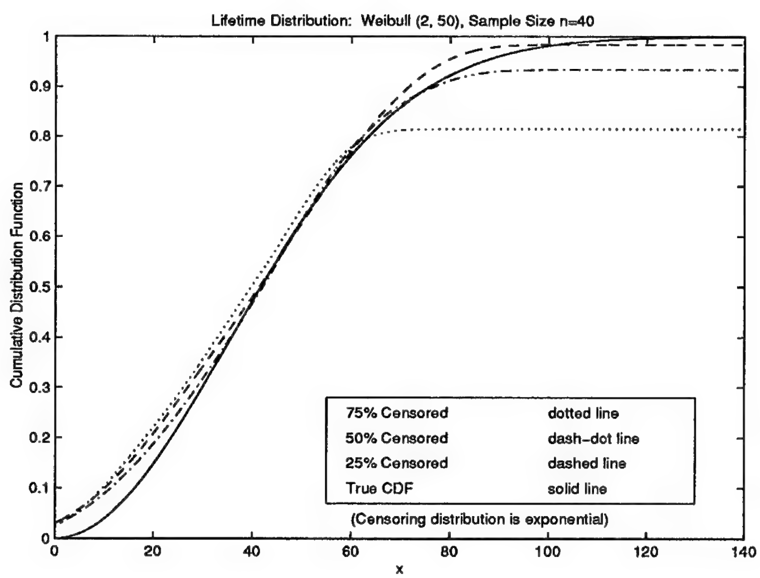

Figure 26 Blum-Susarla Kernel Estimator, Weibull with shape 2.

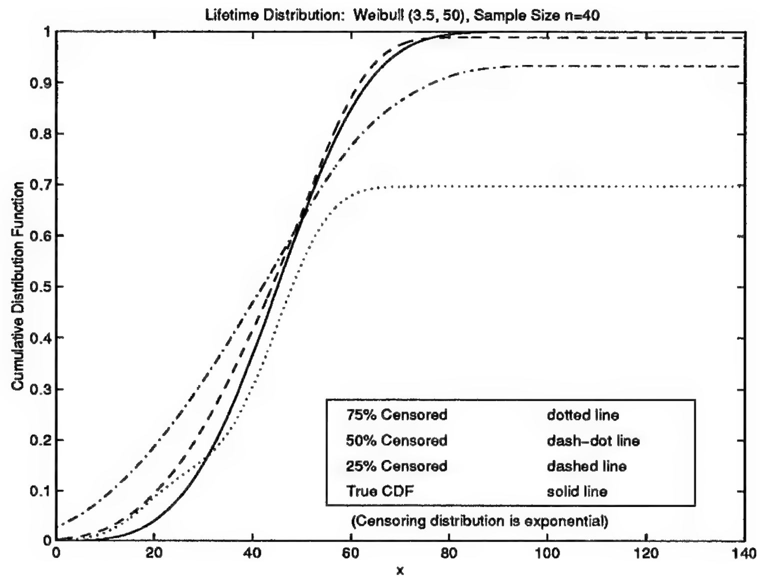

Figure 27 Blum-Susarla Kernel Estimator, Weibull with shape 3.5.

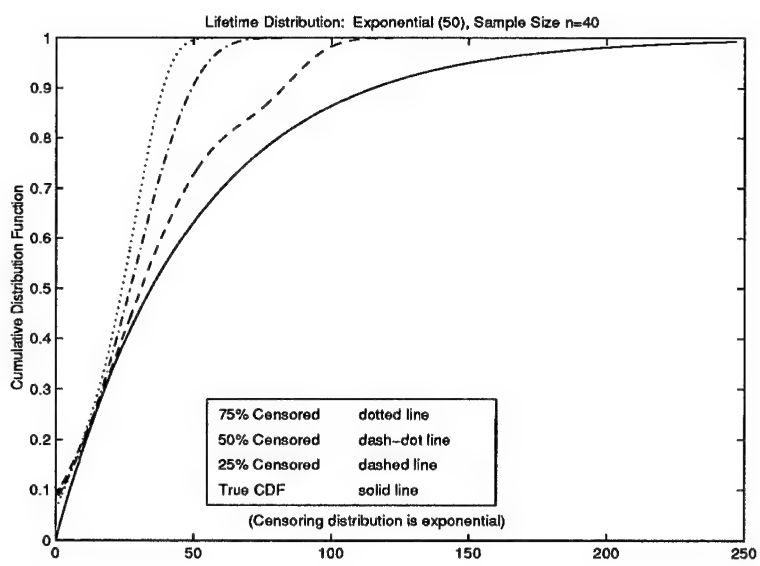

Figure 28 Földes, Rejtő, and Winter Kernel Estimator, Exponential Distribution.

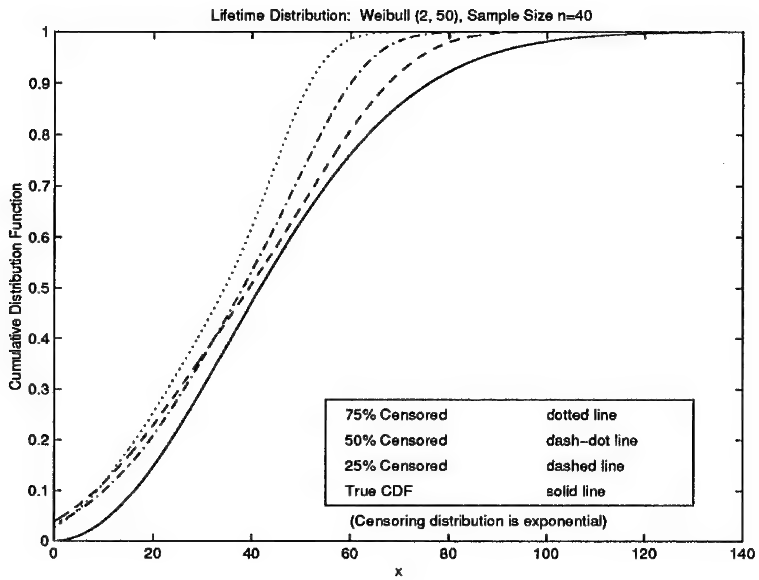

Figure 29 Földes, Rejtő, and Winter Kernel Estimator, Weibull with shape 2.

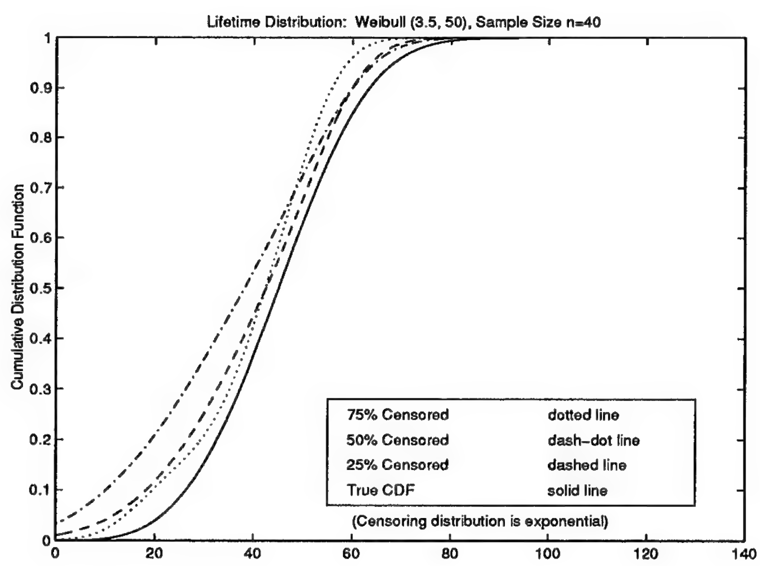

Figure 30 Földes, Rejtő, and Winter Kernel Estimator, Weibull with shape 3.5.

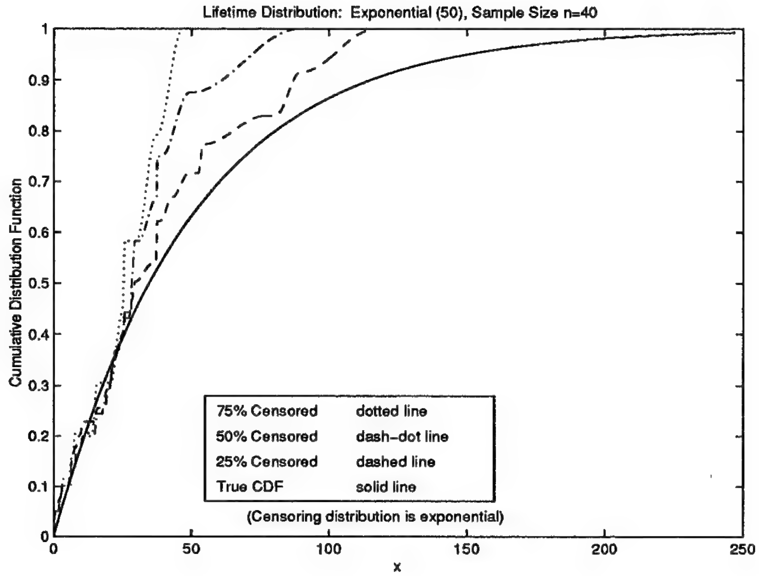

Figure 31 Trigonometrically-Smoothed KME, Exponential Distribution.

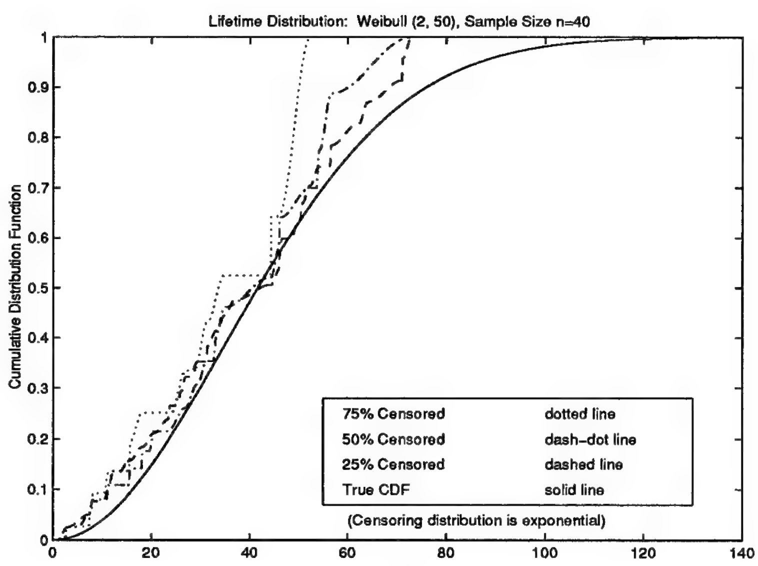

Figure 32 Trigonometrically-Smoothed KME, Weibull with shape 2.

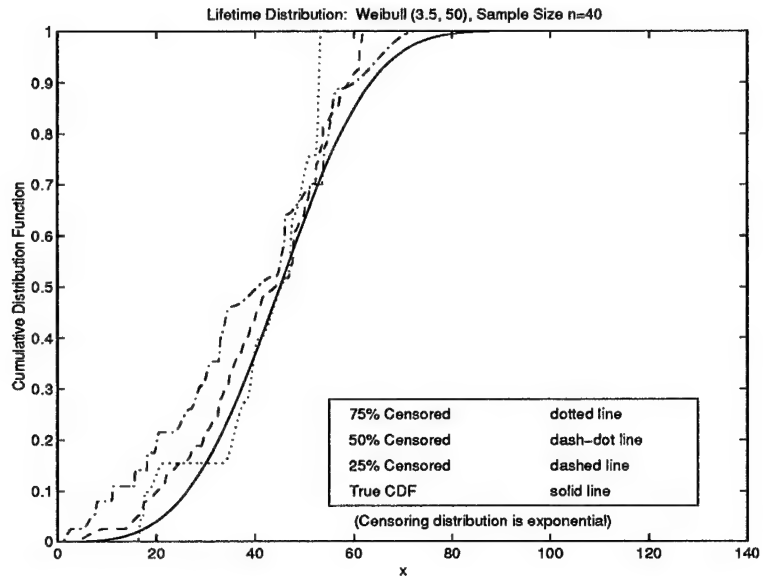

Figure 33 Trigonometrically-Smoothed KME, Weibull with shape 3.5.

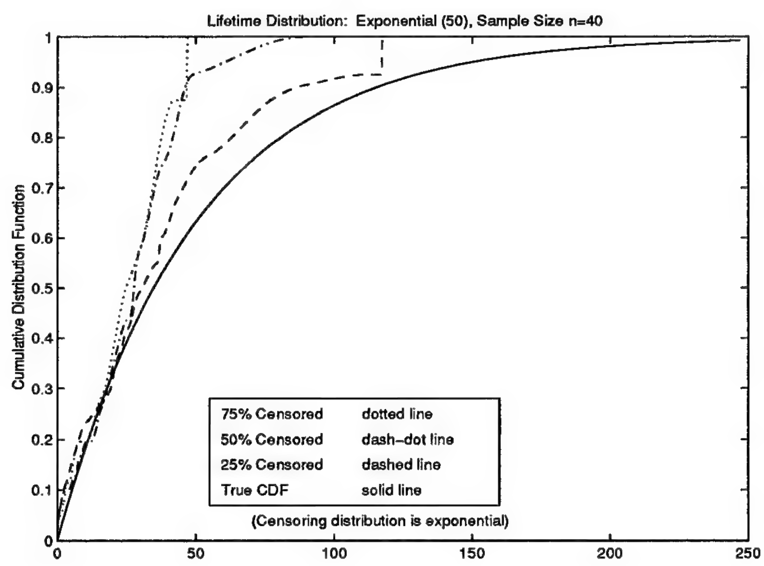

Figure 34 Trigonometrically-Smoothed and Jackknifed KME, Exponential Distribution.

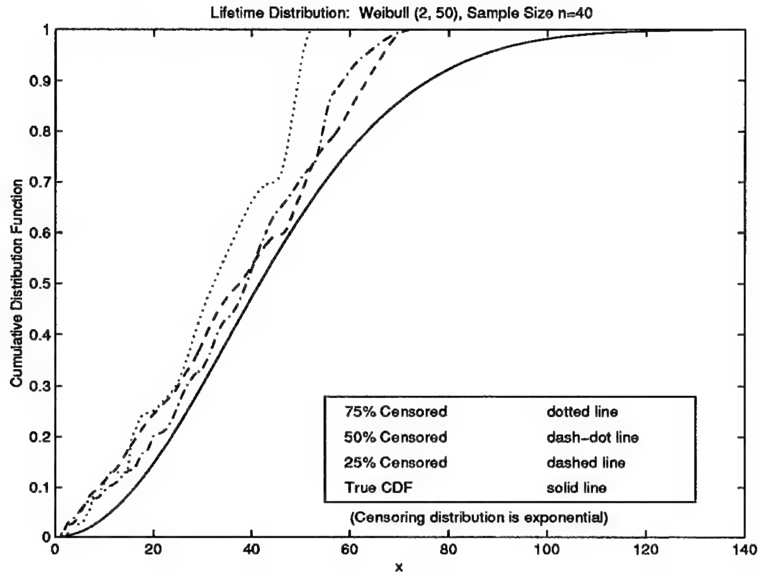

Figure 35 Trigonometrically-Smoothed and Jackknifed KME, Weibull with shape 2.

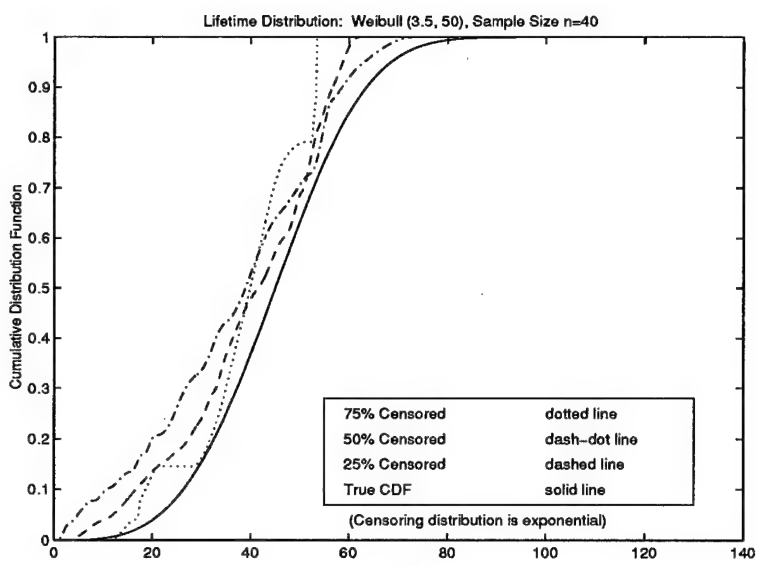

Figure 36 Trigonometrically-Smoothed and Jackknifed KME, Weibull with shape 3.5.

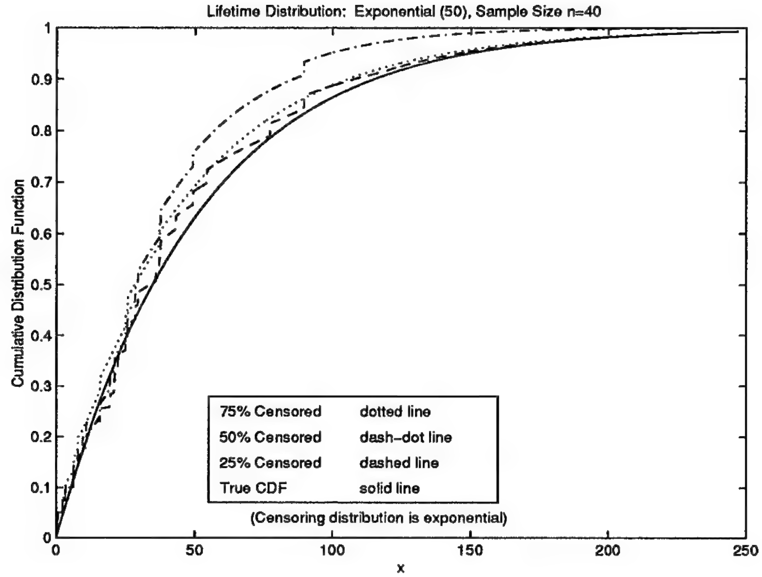

Figure 37 Klein, Lee, and Moeschberger Estimator, Exponential Distribution.

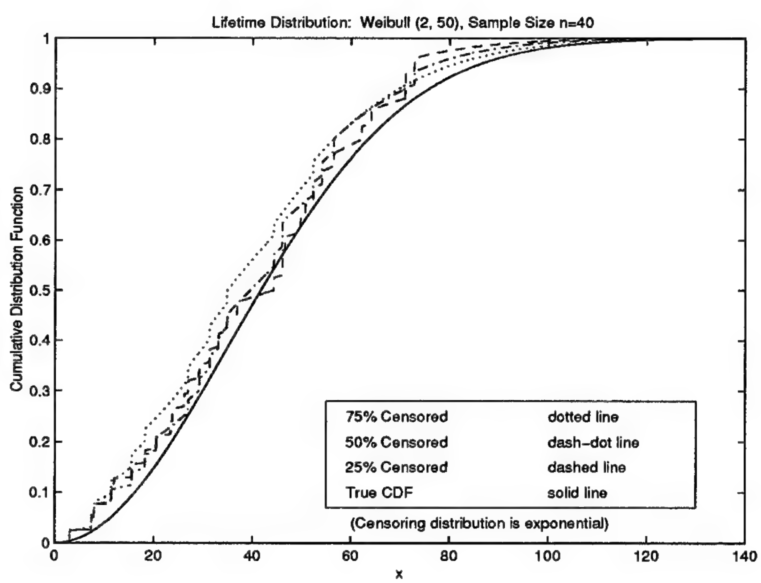

Figure 38 Klein, Lee, and Moeschberger Estimator, Weibull with shape 2.

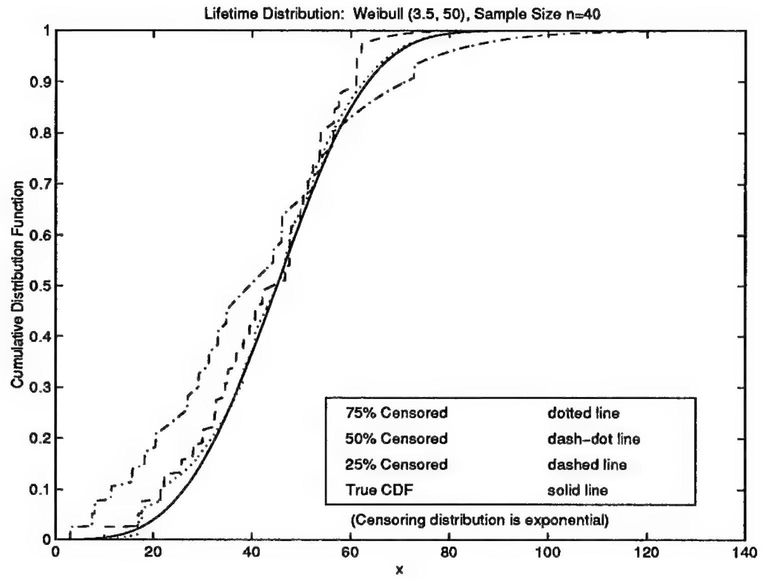

Figure 39 Klein, Lee, and Moeschberger Estimator, Weibull with shape 3.5.

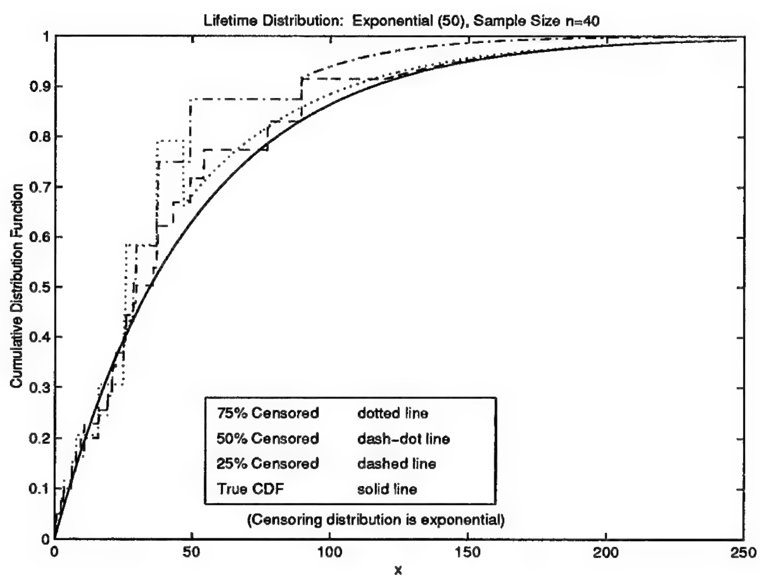

Figure 40 Semi-Parametric Kaplan-Meier Estimator, Exponential Distribution.

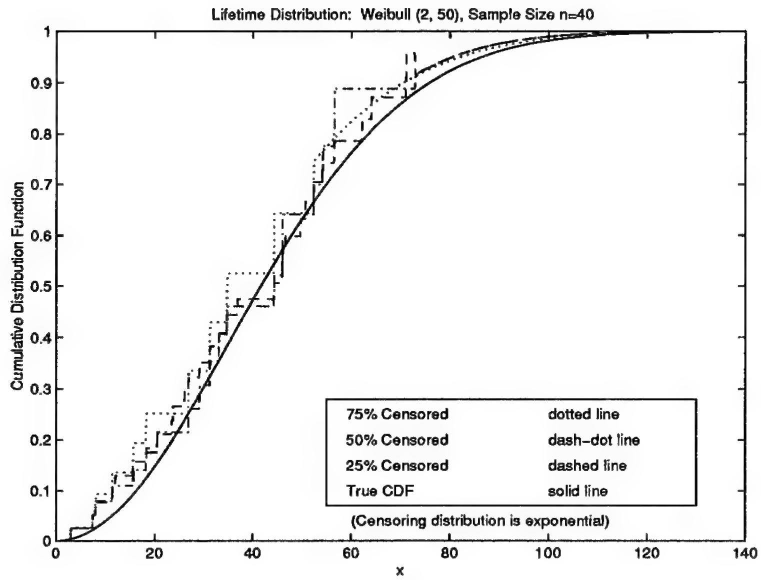

Figure 41 Semi-Parametric Kaplan-Meier Estimator, Weibull with shape 2.

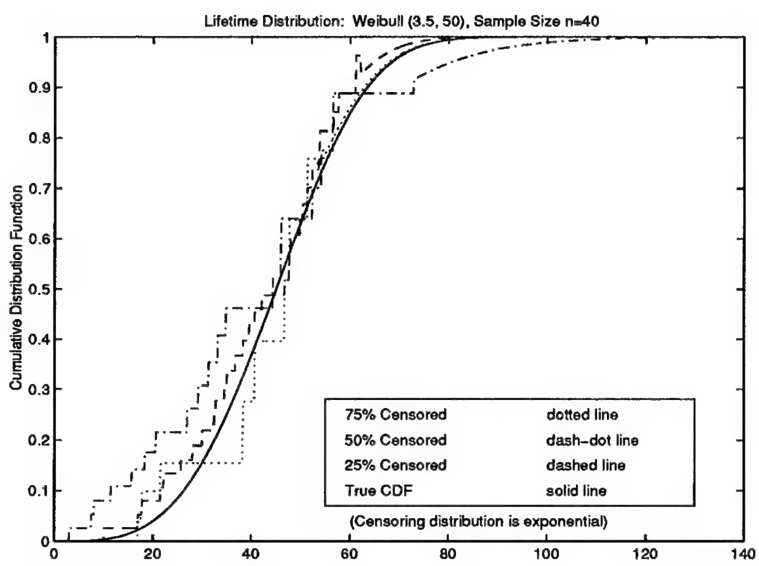

Figure 42 Semi-Parametric Kaplan-Meier Estimator, Weibull with shape 3.5.

Appendix D. Percentage Points for New Tests for the Exponential Distribution

The following tables of percentage points are to be used with the new KME-modified Cramér-von Mises and Anderson-Darling goodness-of-fit tests for exponentiality presented in Section 3.5.

Table 20 Percentage Points of $W^2_{r,n}$ for the Exponential Distribution with Exponential Censoring.

		α					
	n	0.25	0.20	0.15	0.10	0.05	0.025
$q = .10$	20	0.115	0.129	0.146	0.171	0.214	0.258
	40	0.117	0.130	0.147	0.172	0.217	0.262
	60	0.117	0.131	0.148	0.173	0.218	0.263
	80	0.117	0.131	0.148	0.173	0.218	0.264
	100	0.117	0.131	0.148	0.174	0.219	0.265
	120	0.117	0.131	0.149	0.174	0.219	0.265
	140	0.118	0.131	0.149	0.175	0.219	0.265
	160	0.118	0.131	0.149	0.175	0.219	0.265
	180	0.118	0.131	0.149	0.175	0.220	0.266
	200	0.118	0.132	0.149	0.175	0.220	0.266
$q = .20$	20	0.117	0.130	0.147	0.171	0.213	0.256
	40	0.120	0.133	0.150	0.175	0.218	0.261
	60	0.121	0.134	0.151	0.176	0.220	0.264
	80	0.121	0.134	0.151	0.176	0.220	0.264
	100	0.121	0.135	0.151	0.176	0.221	0.265
	120	0.122	0.135	0.152	0.177	0.221	0.266
	140	0.122	0.135	0.153	0.177	0.222	0.267
	160	0.122	0.135	0.153	0.178	0.222	0.268
	180	0.122	0.135	0.153	0.178	0.222	0.268
	200	0.122	0.136	0.153	0.178	0.223	0.268
$q = .30$	20	0.119	0.132	0.149	0.173	0.215	0.258
	40	0.126	0.139	0.156	0.181	0.224	0.269
	60	0.128	0.141	0.159	0.185	0.228	0.273
	80	0.129	0.142	0.160	0.185	0.229	0.275
	100	0.129	0.143	0.161	0.186	0.230	0.277
	120	0.130	0.144	0.161	0.187	0.231	0.278
	140	0.130	0.144	0.162	0.188	0.232	0.279
	160	0.131	0.144	0.162	0.188	0.232	0.279
	180	0.131	0.145	0.163	0.188	0.233	0.280
	200	0.131	0.145	0.163	0.188	0.233	0.280

Table 21 Percentage Points of $W^2_{r,n}$ for the Exponential Distribution with Exponential Censoring (cont'd).

	n	α 0.25	0.20	0.15	0.10	0.05	0.025
$q = .40$	20	0.121	0.134	0.152	0.177	0.222	0.269
	40	0.133	0.148	0.167	0.193	0.240	0.290
	60	0.139	0.154	0.174	0.201	0.248	0.299
	80	0.142	0.157	0.176	0.204	0.252	0.303
	100	0.144	0.159	0.179	0.207	0.256	0.307
	120	0.146	0.161	0.181	0.210	0.258	0.308
	140	0.147	0.162	0.182	0.211	0.260	0.310
	160	0.148	0.163	0.183	0.211	0.261	0.312
	180	0.149	0.165	0.184	0.212	0.262	0.313
	200	0.149	0.165	0.186	0.213	0.263	0.314
$q = .50$	20	0.120	0.134	0.153	0.181	0.231	0.287
	40	0.142	0.158	0.180	0.211	0.267	0.330
	60	0.154	0.171	0.194	0.228	0.288	0.352
	80	0.162	0.180	0.204	0.238	0.299	0.366
	100	0.168	0.187	0.211	0.247	0.310	0.376
	120	0.173	0.192	0.217	0.252	0.316	0.384
	140	0.177	0.196	0.221	0.257	0.322	0.390
	160	0.180	0.199	0.225	0.261	0.326	0.394
	180	0.182	0.201	0.227	0.263	0.329	0.398
	200	0.184	0.204	0.229	0.266	0.331	0.401
$q = .60$	20	0.112	0.128	0.149	0.180	0.240	0.306
	40	0.146	0.165	0.191	0.230	0.306	0.394
	60	0.168	0.189	0.219	0.264	0.349	0.447
	80	0.184	0.208	0.239	0.286	0.376	0.480
	100	0.198	0.222	0.256	0.305	0.400	0.513
	120	0.209	0.235	0.270	0.322	0.422	0.536
	140	0.217	0.245	0.281	0.335	0.437	0.555
	160	0.226	0.254	0.291	0.345	0.450	0.571
	180	0.232	0.262	0.299	0.356	0.462	0.585
	200	0.239	0.267	0.305	0.363	0.472	0.596

Table 22 Percentage Points of $W^2_{r,n}$ for the Exponential Distribution with Exponential Censoring(cont'd).

	n	0.25	0.20	0.15	0.10	0.05	0.025
		α					
$q = .70$	20	0.094	0.111	0.133	0.166	0.233	0.312
	40	0.133	0.155	0.185	0.233	0.327	0.441
	60	0.165	0.191	0.228	0.284	0.400	0.543
	80	0.191	0.221	0.263	0.329	0.462	0.626
	100	0.214	0.248	0.295	0.367	0.513	0.689
	120	0.235	0.272	0.323	0.402	0.560	0.755
	140	0.253	0.292	0.346	0.429	0.596	0.805
	160	0.271	0.312	0.370	0.457	0.633	0.851
	180	0.285	0.329	0.388	0.481	0.668	0.891
	200	0.298	0.344	0.406	0.502	0.692	0.924
$q = .80$	20	0.061	0.076	0.098	0.130	0.195	0.276
	40	0.089	0.109	0.140	0.187	0.283	0.409
	60	0.119	0.146	0.185	0.247	0.377	0.550
	80	0.148	0.180	0.228	0.303	0.461	0.671
	100	0.175	0.213	0.267	0.355	0.539	0.777
	120	0.200	0.243	0.304	0.401	0.608	0.878
	140	0.224	0.271	0.338	0.448	0.674	0.966
	160	0.247	0.299	0.371	0.487	0.734	1.057
	180	0.269	0.324	0.402	0.526	0.789	1.127
	200	0.290	0.348	0.430	0.564	0.851	1.203
$q = .90$	20	0.028	0.038	0.052	0.076	0.131	0.195
	40	0.027	0.036	0.051	0.079	0.141	0.233
	60	0.035	0.046	0.065	0.099	0.182	0.299
	80	0.045	0.059	0.083	0.126	0.228	0.374
	100	0.055	0.073	0.101	0.152	0.271	0.445
	120	0.066	0.087	0.120	0.178	0.317	0.513
	140	0.077	0.101	0.139	0.206	0.365	0.588
	160	0.088	0.115	0.157	0.232	0.410	0.656
	180	0.099	0.129	0.176	0.259	0.456	0.734
	200	0.111	0.143	0.194	0.285	0.496	0.795

Table 23 Percentage Points of $A^2_{r,n}$ for the Exponential Distribution with Exponential Censoring.

		α					
	n	0.25	0.20	0.15	0.10	0.05	0.025
$q = .10$	20	0.709	0.784	0.882	1.024	1.274	1.542
	40	0.733	0.811	0.911	1.053	1.310	1.571
	60	0.745	0.822	0.923	1.069	1.323	1.593
	80	0.750	0.827	0.926	1.071	1.329	1.591
	100	0.753	0.832	0.932	1.078	1.336	1.603
	120	0.759	0.836	0.937	1.083	1.340	1.608
	140	0.762	0.840	0.941	1.086	1.341	1.610
	160	0.761	0.840	0.941	1.089	1.343	1.612
	180	0.763	0.841	0.944	1.091	1.348	1.615
	200	0.765	0.843	0.945	1.091	1.348	1.609
$q = .20$	20	0.722	0.800	0.901	1.044	1.309	1.587
	40	0.765	0.844	0.948	1.099	1.364	1.650
	60	0.785	0.865	0.970	1.119	1.389	1.668
	80	0.795	0.875	0.980	1.131	1.399	1.674
	100	0.803	0.884	0.989	1.140	1.408	1.685
	120	0.811	0.893	0.997	1.149	1.419	1.692
	140	0.815	0.896	1.002	1.154	1.421	1.706
	160	0.819	0.900	1.007	1.158	1.427	1.708
	180	0.822	0.904	1.010	1.162	1.430	1.705
	200	0.824	0.905	1.014	1.166	1.433	1.708
$q = .30$	20	0.730	0.808	0.913	1.065	1.341	1.642
	40	0.799	0.883	0.995	1.156	1.451	1.782
	60	0.834	0.923	1.038	1.205	1.504	1.838
	80	0.857	0.946	1.061	1.228	1.534	1.861
	100	0.875	0.964	1.082	1.251	1.558	1.894
	120	0.889	0.979	1.097	1.267	1.574	1.906
	140	0.900	0.993	1.112	1.284	1.595	1.927
	160	0.910	1.001	1.120	1.291	1.601	1.933
	180	0.918	1.011	1.131	1.303	1.614	1.940
	200	0.922	1.015	1.135	1.309	1.620	1.949

Table 24 Percentage Points of $A^2_{r,n}$ for the Exponential Distribution with Exponential Censoring (cont'd).

	n	α 0.25	0.20	0.15	0.10	0.05	0.025
$q = .40$	20	0.725	0.807	0.915	1.075	1.374	1.729
	40	0.829	0.922	1.043	1.225	1.572	1.978
	60	0.892	0.991	1.121	1.317	1.682	2.109
	80	0.932	1.033	1.170	1.370	1.750	2.191
	100	0.966	1.070	1.210	1.416	1.804	2.254
	120	0.994	1.101	1.244	1.456	1.845	2.312
	140	1.016	1.125	1.268	1.482	1.885	2.343
	160	1.035	1.146	1.291	1.508	1.916	2.375
	180	1.052	1.164	1.314	1.531	1.946	2.415
	200	1.064	1.177	1.327	1.547	1.966	2.432
$q = .50$	20	0.701	0.784	0.895	1.060	1.386	1.785
	40	0.842	0.941	1.076	1.282	1.689	2.195
	60	0.933	1.045	1.196	1.428	1.881	2.464
	80	1.002	1.121	1.284	1.527	2.027	2.661
	100	1.062	1.187	1.360	1.620	2.139	2.803
	120	1.112	1.243	1.423	1.695	2.248	2.927
	140	1.150	1.288	1.472	1.759	2.332	3.035
	160	1.190	1.329	1.523	1.815	2.392	3.119
	180	1.223	1.366	1.561	1.860	2.447	3.199
	200	1.246	1.395	1.596	1.903	2.516	3.285
$q = .60$	20	0.644	0.727	0.839	1.009	1.351	1.771
	40	0.817	0.922	1.068	1.294	1.763	2.379
	60	0.938	1.064	1.236	1.502	2.069	2.812
	80	1.039	1.176	1.365	1.662	2.292	3.147
	100	1.128	1.276	1.481	1.808	2.510	3.451
	120	1.201	1.362	1.588	1.941	2.693	3.703
	140	1.268	1.438	1.675	2.046	2.848	3.918
	160	1.330	1.510	1.757	2.147	2.978	4.103
	180	1.386	1.574	1.835	2.243	3.114	4.310
	200	1.435	1.625	1.891	2.313	3.228	4.455

Table 25 Percentage Points of $A^2_{r,n}$ for the Exponential Distribution with Exponential Censoring (cont'd).

	n	α 0.25	0.20	0.15	0.10	0.05	0.025
$q = .70$	20	0.544	0.622	0.728	0.889	1.218	1.634
	40	0.717	0.824	0.969	1.206	1.685	2.316
	60	0.862	0.992	1.172	1.457	2.068	2.892
	80	0.986	1.135	1.345	1.681	2.407	3.398
	100	1.099	1.268	1.506	1.879	2.702	3.790
	120	1.204	1.387	1.652	2.078	2.988	4.240
	140	1.295	1.494	1.774	2.229	3.201	4.565
	160	1.386	1.600	1.908	2.389	3.441	4.890
	180	1.464	1.694	2.016	2.533	3.660	5.211
	200	1.539	1.780	2.118	2.664	3.856	5.462
$q = .80$	20	0.393	0.463	0.557	0.703	0.994	1.351
	40	0.518	0.610	0.738	0.940	1.369	1.927
	60	0.644	0.758	0.926	1.195	1.763	2.546
	80	0.764	0.902	1.105	1.428	2.120	3.067
	100	0.876	1.039	1.272	1.652	2.466	3.563
	120	0.980	1.163	1.424	1.846	2.767	4.009
	140	1.077	1.280	1.569	2.049	3.061	4.424
	160	1.174	1.395	1.708	2.221	3.325	4.846
	180	1.266	1.502	1.840	2.387	3.577	5.182
	200	1.353	1.606	1.964	2.556	3.851	5.556
$q = .90$	20	0.243	0.292	0.363	0.476	0.709	0.990
	40	0.241	0.294	0.370	0.499	0.782	1.171
	60	0.285	0.347	0.442	0.602	0.969	1.464
	80	0.337	0.412	0.527	0.724	1.177	1.795
	100	0.392	0.480	0.615	0.848	1.371	2.097
	120	0.446	0.547	0.702	0.964	1.571	2.386
	140	0.498	0.613	0.788	1.085	1.780	2.713
	160	0.551	0.677	0.870	1.200	1.968	2.995
	180	0.602	0.742	0.956	1.319	2.161	3.321
	200	0.656	0.808	1.036	1.432	2.331	3.573

Appendix E. Percentage Points for New Tests for the Weibull Distribution

The following tables of percentage points are to be used with the new KME-modified Cramér-von Mises and Anderson-Darling goodness-of-fit tests for the Weibull distribution with shape parameters 2 and 3.5 presented in Section 3.6.

Table 26 Percentage Points of $W^2_{r,n}$ for the Weibull Distribution with Shape $\beta = 2$ and Exponential Censoring.

	n	0.25	0.20	0.15	0.10	0.05	0.025
		α					
$q = .10$	20	0.116	0.128	0.143	0.165	0.201	0.237
	40	0.115	0.127	0.143	0.164	0.202	0.238
	60	0.114	0.126	0.141	0.163	0.200	0.238
	80	0.113	0.125	0.140	0.162	0.199	0.236
	100	0.112	0.124	0.139	0.160	0.197	0.235
$q = .20$	20	0.121	0.133	0.149	0.171	0.208	0.245
	40	0.119	0.131	0.147	0.168	0.206	0.243
	60	0.117	0.129	0.145	0.167	0.204	0.241
	80	0.116	0.128	0.143	0.165	0.202	0.239
	100	0.115	0.127	0.142	0.164	0.201	0.238
$q = .30$	20	0.130	0.143	0.160	0.184	0.226	0.266
	40	0.129	0.142	0.159	0.182	0.222	0.262
	60	0.128	0.141	0.157	0.181	0.220	0.261
	80	0.127	0.140	0.156	0.180	0.219	0.260
	100	0.126	0.139	0.156	0.179	0.218	0.260
$q = .40$	20	0.144	0.159	0.179	0.207	0.255	0.305
	40	0.146	0.161	0.180	0.207	0.253	0.300
	60	0.146	0.161	0.180	0.206	0.253	0.299
	80	0.146	0.160	0.179	0.205	0.250	0.296
	100	0.145	0.160	0.179	0.205	0.251	0.298
$q = .50$	20	0.163	0.182	0.205	0.239	0.301	0.368
	40	0.172	0.190	0.213	0.246	0.304	0.365
	60	0.174	0.192	0.215	0.248	0.305	0.364
	80	0.175	0.192	0.215	0.248	0.305	0.361
	100	0.175	0.193	0.216	0.248	0.304	0.360
$q = .60$	20	0.185	0.208	0.239	0.284	0.369	0.469
	40	0.209	0.232	0.263	0.307	0.389	0.478
	60	0.218	0.242	0.273	0.318	0.397	0.481
	80	0.223	0.247	0.277	0.322	0.399	0.481
	100	0.227	0.250	0.281	0.325	0.401	0.479
$q = .70$	20	0.202	0.233	0.273	0.336	0.465	0.631
	40	0.254	0.288	0.334	0.403	0.539	0.705
	60	0.285	0.321	0.369	0.441	0.576	0.737
	80	0.305	0.341	0.390	0.461	0.596	0.751
	100	0.318	0.355	0.404	0.476	0.612	0.760
$q = .80$	20	0.199	0.237	0.294	0.384	0.578	0.833
	40	0.281	0.332	0.403	0.517	0.770	1.104
	60	.356	0.413	0.495	0.629	0.908	1.274
	80	0.413	0.478	0.567	0.710	1.002	1.384
	100	0.461	0.531	0.624	0.772	1.077	1.474
$q = .90$	20	0.157	0.204	0.270	0.388	0.676	1.052
	40	0.173	0.233	0.322	0.475	0.858	1.440
	60	0.253	0.328	0.442	0.645	1.153	1.927
	80	0.350	0.444	0.587	0.839	1.450	2.374
	100	0.453	0.566	0.736	1.035	1.745	2.787

Table 27 Percentage Points of $W^2_{r,n}$ for the Weibull Distribution with Shape $\beta = 3.5$ and Exponential Censoring.

	n	α 0.25	0.20	0.15	0.10	0.05	0.025
$q = .10$	20	0.214	0.245	0.283	0.338	0.426	0.509
	40	0.199	0.230	0.271	0.331	0.434	0.535
	60	0.184	0.213	0.253	0.312	0.417	0.522
	80	0.171	0.197	0.233	0.289	0.391	0.500
	100	0.162	0.187	0.220	0.271	0.366	0.472
$q = .20$	20	0.169	0.191	0.221	0.263	0.337	0.413
	40	0.152	0.171	0.197	0.234	0.302	0.377
	60	0.144	0.161	0.184	0.218	0.279	0.345
	80	0.139	0.155	0.176	0.207	0.263	0.323
	100	0.136	0.151	0.171	0.201	0.253	0.309
$q = .30$	20	0.162	0.180	0.205	0.240	0.301	0.366
	40	0.151	0.168	0.190	0.220	0.276	0.333
	60	0.146	0.162	0.183	0.212	0.264	0.318
	80	0.144	0.159	0.179	0.208	0.258	0.309
	100	0.142	0.157	0.177	0.205	0.254	0.304
$q = .40$	20	0.171	0.190	0.214	0.248	0.310	0.371
	40	0.165	0.183	0.205	0.237	0.293	0.350
	60	0.162	0.179	0.201	0.233	0.288	0.344
	80	0.161	0.177	0.199	0.230	0.284	0.339
	100	0.159	0.176	0.198	0.229	0.282	0.337
$q = .50$	20	0.192	0.214	0.241	0.281	0.351	0.422
	40	0.191	0.212	0.238	0.275	0.339	0.404
	60	0.190	0.210	0.236	0.273	0.336	0.401
	80	0.189	0.209	0.234	0.270	0.334	0.397
	100	0.189	0.208	0.234	0.270	0.331	0.395
$q = .60$	20	0.226	0.252	0.287	0.336	0.426	0.522
	40	0.233	0.259	0.291	0.339	0.420	0.503
	60	0.235	0.260	0.292	0.337	0.416	0.497
	80	0.235	0.260	0.291	0.336	0.414	0.494
	100	0.235	0.260	0.291	0.336	0.414	0.490
$q = .70$	20	0.272	0.307	0.354	0.424	0.556	0.714
	40	0.299	0.334	0.379	0.445	0.566	0.701
	60	0.310	0.344	0.389	0.453	0.568	0.690
	80	0.315	0.349	0.394	0.457	0.568	0.683
	100	0.318	0.352	0.396	0.458	0.562	0.678
$q = .80$	20	0.334	0.383	0.451	0.558	0.773	1.036
	40	0.401	0.456	0.529	0.641	0.865	1.142
	60	0.445	0.501	0.576	0.688	0.900	1.154
	80	0.468	0.524	0.599	0.708	0.916	1.157
	100	0.485	0.540	0.615	0.723	0.922	1.146
$q = .90$	20	0.421	0.497	0.608	0.786	1.127	1.506
	40	0.523	0.623	0.767	1.005	1.528	2.206
	60	0.634	0.748	0.914	1.177	1.786	2.582
	80	0.737	0.864	1.044	1.329	1.976	2.839
	100	0.824	0.956	1.145	1.445	2.094	2.940

Table 28 Percentage Points of $A^2_{r,n}$ for the Weibull Distribution with Shape $\beta = 2$ and Exponential Censoring.

	n	α 0.25	0.20	0.15	0.10	0.05	0.025
$q = .10$	20	0.799	0.866	0.949	1.062	1.255	1.442
	40	0.793	0.861	0.949	1.069	1.267	1.464
	60	0.781	0.851	0.939	1.061	1.261	1.466
	80	0.774	0.838	0.926	1.049	1.255	1.452
	100	0.761	0.831	0.919	1.039	1.242	1.441
$q = .20$	20	0.807	0.877	0.967	1.091	1.301	1.504
	40	0.794	0.865	0.954	1.076	1.286	1.496
	60	0.782	0.852	0.941	1.064	1.270	1.476
	80	0.772	0.841	0.928	1.048	1.253	1.457
	100	0.765	0.832	0.918	1.038	1.243	1.443
$q = .30$	20	0.842	0.918	1.019	1.159	1.404	1.656
	40	0.841	0.918	1.016	1.154	1.388	1.626
	60	0.837	0.912	1.009	1.142	1.374	1.612
	80	0.834	0.909	1.001	1.134	1.362	1.591
	100	0.830	0.905	0.997	1.126	1.355	1.583
$q = .40$	20	0.903	0.991	1.107	1.273	1.582	1.920
	40	0.931	1.021	1.135	1.298	1.588	1.908
	60	0.943	1.032	1.146	1.307	1.594	1.888
	80	0.948	1.036	1.148	1.306	1.584	1.873
	100	0.952	1.039	1.151	1.308	1.584	1.872
$q = .50$	20	0.983	1.089	1.233	1.443	1.856	2.356
	40	1.063	1.174	1.318	1.532	1.930	2.391
	60	1.104	1.216	1.360	1.573	1.969	2.413
	80	1.123	1.233	1.380	1.593	1.979	2.407
	100	1.144	1.254	1.399	1.608	1.985	2.395
$q = .60$	20	1.071	1.203	1.383	1.661	2.238	2.989
	40	1.241	1.386	1.579	1.875	2.478	3.294
	60	1.335	1.486	1.689	1.995	2.601	3.382
	80	1.394	1.551	1.757	2.068	2.670	3.400
	100	1.441	1.599	1.810	2.124	2.725	3.429
$q = .70$	20	1.116	1.282	1.512	1.885	2.726	3.907
	40	1.429	1.629	1.912	2.364	3.379	4.838
	60	1.639	1.863	2.176	2.677	3.781	5.296
	80	1.791	2.026	2.346	2.870	4.008	5.572
	100	1.903	2.144	2.487	3.035	4.189	5.733
$q = .80$	20	1.060	1.256	1.538	2.015	3.142	4.706
	40	1.473	1.742	2.127	2.797	4.456	6.825
	60	1.868	2.189	2.668	3.492	5.483	8.400
	80	2.192	2.558	3.102	4.032	6.271	9.589
	100	2.470	2.875	3.458	4.478	6.947	10.508
$q = .90$	20	0.848	1.078	1.383	1.933	3.325	5.294
	40	0.969	1.235	1.636	2.355	4.289	7.503
	60	1.343	1.685	2.200	3.163	5.788	10.352
	80	1.799	2.223	2.885	4.094	7.403	13.014
	100	2.280	2.787	3.594	5.072	9.019	15.535

Table 29 Percentage Points of $A^2_{r,n}$ for the Weibull Distribution with Shape $\beta = 3.5$ and Exponential Censoring.

		α					
	n	0.25	0.20	0.15	0.10	0.05	0.025
$q = .10$	20	1.956	2.214	2.523	2.931	3.550	4.095
	40	1.648	1.920	2.275	2.772	3.579	4.328
	60	1.414	1.661	2.007	2.507	3.368	4.210
	80	1.258	1.473	1.775	2.231	3.074	3.932
	100	1.163	1.349	1.612	2.028	2.809	3.643
$q = .20$	20	1.343	1.542	1.804	2.182	2.808	3.399
	40	1.079	1.222	1.418	1.719	2.288	2.910
	60	0.982	1.101	1.260	1.503	1.967	2.487
	80	0.930	1.035	1.176	1.383	1.775	2.213
	100	0.901	0.998	1.128	1.318	1.664	2.046
$q = .30$	20	1.146	1.288	1.483	1.773	2.297	2.842
	40	1.006	1.116	1.262	1.478	1.862	2.285
	60	0.961	1.060	1.191	1.377	1.714	2.063
	80	0.939	1.031	1.151	1.324	1.633	1.951
	100	0.925	1.015	1.133	1.298	1.584	1.893
$q = .40$	20	1.128	1.258	1.428	1.682	2.140	2.627
	40	1.065	1.174	1.316	1.521	1.880	2.262
	60	1.044	1.146	1.279	1.473	1.801	2.150
	80	1.034	1.134	1.263	1.447	1.761	2.091
	100	1.029	1.128	1.253	1.431	1.740	2.060
$q = .50$	20	1.209	1.343	1.524	1.793	2.279	2.836
	40	1.205	1.330	1.489	1.721	2.137	2.583
	60	1.204	1.327	1.482	1.703	2.098	2.511
	80	1.205	1.322	1.474	1.694	2.074	2.463
	100	1.210	1.327	1.474	1.688	2.055	2.434
$q = .60$	20	1.356	1.515	1.731	2.053	2.674	3.437
	40	1.431	1.584	1.788	2.083	2.637	3.273
	60	1.460	1.615	1.811	2.098	2.612	3.183
	80	1.478	1.628	1.825	2.104	2.610	3.148
	100	1.495	1.643	1.836	2.114	2.598	3.122
$q = .70$	20	1.566	1.770	2.050	2.482	3.373	4.601
	40	1.764	1.972	2.253	2.691	3.570	4.710
	60	1.871	2.087	2.371	2.791	3.616	4.617
	80	1.934	2.147	2.430	2.846	3.635	4.572
	100	1.979	2.188	2.470	2.880	3.651	4.519
$q = .80$	20	1.833	2.102	2.485	3.106	4.469	6.233
	40	2.248	2.567	3.009	3.737	5.339	7.622
	60	2.545	2.887	3.363	4.124	5.769	8.046
	80	2.723	3.073	3.562	4.331	5.972	8.179
	100	2.871	3.226	3.719	4.490	6.076	8.210
$q = .90$	20	2.207	2.590	3.161	4.095	6.109	8.596
	40	2.753	3.263	4.030	5.363	8.567	12.979
	60	3.333	3.951	4.877	6.490	10.494	16.435
	80	3.893	4.593	5.642	7.513	12.110	18.985
	100	4.382	5.155	6.2947	8.308	13.222	20.517

Table 30 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.10$ and $q = 0.20$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.107	0.895	0.259	0.512	0.208
	$A^2_{r,n}$	0.108	0.860	0.210	0.457	0.186
	B^2	0.108	0.566	0.064	0.193	0.143
	$n(1 - R^2_n)$	0.061	0.402	0.093	0.162	0.042
	STCL-CvM	0.078	0.802	0.206	0.428	0.155
	STCL-AD	0.091	0.774	0.188	0.386	0.157
$n = 40$	$W^2_{r,n}$	0.084	0.998	0.409	0.832	0.310
	$A^2_{r,n}$	0.087	0.997	0.384	0.828	0.340
	B^2	0.093	0.984	0.229	0.644	0.239
	$n(1 - R^2_n)$	0.062	0.877	0.189	0.411	0.098
	STCL-CvM	0.076	0.994	0.337	0.736	0.270
	STCL-AD	0.085	0.993	0.337	0.743	0.293
$n = 60$	$W^2_{r,n}$	0.100	1.000	0.552	0.944	0.435
	$A^2_{r,n}$	0.106	1.000	0.549	0.952	0.534
	B^2	0.088	1.000	0.391	0.878	0.363
	$n(1 - R^2_n)$	0.079	0.990	0.266	0.640	0.104
	STCL-CvM	0.104	1.000	0.458	0.901	0.335
	STCL-AD	0.096	1.000	0.457	0.907	0.422
$n = 80$	$W^2_{r,n}$	0.100	1.000	0.700	0.991	0.556
	$A^2_{r,n}$	0.094	1.000	0.704	0.995	0.670
	B^2	0.088	1.000	0.576	0.978	0.510
	$n(1 - R^2_n)$	0.087	0.998	0.369	0.771	0.148
	STCL-CvM	0.099	1.000	0.599	0.975	0.457
	STCL-AD	0.117	1.000	0.600	0.988	0.587
$n = 100$	$W^2_{r,n}$	0.100	1.000	0.796	0.997	0.622
	$A^2_{r,n}$	0.104	1.000	0.809	0.998	0.743
	B^2	0.119	1.000	0.708	0.994	0.661
	$n(1 - R^2_n)$	0.077	1.000	0.432	0.845	0.157
	STCL-CvM	0.101	1.000	0.705	0.994	0.569
	STCL-AD	0.101	1.000	0.722	0.998	0.731

Table 31 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.10$ and $q = 0.50$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.099	0.544	0.160	0.268	0.135
	$A^2_{r,n}$	0.105	0.512	0.132	0.245	0.124
	B^2	0.092	0.280	0.043	0.103	0.044
	STCL-CvM	0.093	0.530	0.174	0.284	0.191
	STCL-AD	0.096	0.467	0.141	0.240	0.157
$n = 40$	$W^2_{r,n}$	0.103	0.888	0.213	0.493	0.198
	$A^2_{r,n}$	0.101	0.876	0.194	0.458	0.208
	B^2	0.102	0.853	0.169	0.464	0.197
	STCL-CvM	0.076	0.860	0.240	0.518	0.334
	STCL-AD	0.089	0.833	0.221	0.490	0.329
$n = 60$	$W^2_{r,n}$	0.120	0.974	0.279	0.697	0.264
	$A^2_{r,n}$	0.111	0.973	0.250	0.690	0.258
	B^2	0.098	0.982	0.307	0.765	0.411
	STCL-CvM	0.086	0.960	0.306	0.703	0.434
	STCL-AD	0.094	0.957	0.293	0.699	0.468
$n = 80$	$W^2_{r,n}$	0.109	1.000	0.379	0.837	0.356
	$A^2_{r,n}$	0.098	1.000	0.348	0.836	0.386
	B^2	0.094	0.998	0.467	0.914	0.617
	STCL-CvM	0.081	0.995	0.410	0.822	0.586
	STCL-AD	0.080	0.994	0.390	0.836	0.659
$n = 100$	$W^2_{r,n}$	0.095	1.000	0.492	0.916	0.424
	$A^2_{r,n}$	0.096	1.000	0.463	0.917	0.448
	B^2	0.121	1.000	0.605	0.970	0.780
	STCL-CvM	0.096	1.000	0.497	0.905	0.701
	STCL-AD	0.092	1.000	0.500	0.928	0.779

Table 32 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.10$ and $q = 0.80$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.121	0.227	0.113	0.158	0.113
	$A^2_{r,n}$	0.124	0.264	0.111	0.178	0.127
	B^2	0.092	0.003	0.022	0.010	0.002
	STCL-CvM	0.100	0.197	0.103	0.168	0.176
	STCL-AD	0.110	0.155	0.080	0.132	0.137
$n = 40$	$W^2_{r,n}$	0.104	0.366	0.131	0.205	0.122
	$A^2_{r,n}$	0.106	0.441	0.142	0.245	0.154
	B^2	0.097	0.038	0.017	0.021	0.014
	STCL-CvM	0.100	0.336	0.155	0.245	0.253
	STCL-AD	0.102	0.275	0.132	0.202	0.204
$n = 60$	$W^2_{r,n}$	0.120	0.974	0.279	0.697	0.264
	$A^2_{r,n}$	0.111	0.973	0.250	0.690	0.258
	B^2	0.108	0.185	0.045	0.098	0.108
	STCL-CvM	0.091	0.492	0.175	0.313	0.364
	STCL-AD	0.108	0.442	0.149	0.276	0.338
$n = 80$	$W^2_{r,n}$	0.098	0.622	0.173	0.309	0.179
	$A^2_{r,n}$	0.099	0.698	0.187	0.392	0.262
	B^2	0.113	0.381	0.097	0.273	0.262
	STCL-CvM	0.111	0.646	0.201	0.424	0.475
	STCL-AD	0.104	0.600	0.187	0.406	0.448
$n = 100$	$W^2_{r,n}$	0.095	0.745	0.195	0.418	0.199
	$A^2_{r,n}$	0.097	0.811	0.214	0.486	0.269
	B^2	0.099	0.602	0.160	0.454	0.505
	STCL-CvM	0.108	0.745	0.224	0.504	0.604
	STCL-AD	0.097	0.720	0.208	0.493	0.592

Table 33 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.05$ and $q = 0.20$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.053	0.807	0.148	0.381	0.122
	$A^2_{r,n}$	0.058	0.721	0.111	0.296	0.105
	B^2	0.049	0.379	0.022	0.090	0.070
	$n(1 - R^2_n)$	0.027	0.183	0.023	0.057	0.017
	STCL-CvM	0.034	0.680	0.113	0.272	0.085
	STCL-AD	0.043	0.629	0.096	0.231	0.091
$n = 40$	$W^2_{r,n}$	0.041	0.990	0.305	0.727	0.171
	$A^2_{r,n}$	0.041	0.987	0.265	0.695	0.187
	B^2	0.037	0.959	0.127	0.473	0.113
	$n(1 - R^2_n)$	0.024	0.730	0.087	0.244	0.039
	STCL-CvM	0.031	0.977	0.219	0.604	0.148
	STCL-AD	0.033	0.976	0.213	0.599	0.177
$n = 60$	$W^2_{r,n}$	0.058	0.999	0.409	0.885	0.274
	$A^2_{r,n}$	0.050	1.000	0.387	0.892	0.335
	B^2	0.044	0.999	0.247	0.780	0.213
	$n(1 - R^2_n)$	0.028	0.960	0.144	0.447	0.054
	STCL-CvM	0.057	0.999	0.308	0.798	0.218
	STCL-AD	0.055	0.999	0.309	0.831	0.271
$n = 80$	$W^2_{r,n}$	0.052	1.000	0.564	0.965	0.365
	$A^2_{r,n}$	0.051	1.000	0.566	0.978	0.480
	B^2	0.040	1.000	0.414	0.939	0.313
	$n(1 - R^2_n)$	0.036	0.992	0.223	0.610	0.080
	STCL-CvM	0.063	1.000	0.442	0.935	0.298
	STCL-AD	0.063	1.000	0.473	0.951	0.417
$n = 100$	$W^2_{r,n}$	0.047	1.000	0.681	0.993	0.455
	$A^2_{r,n}$	0.047	1.000	0.693	0.996	0.576
	B^2	0.045	1.000	0.570	0.981	0.442
	$n(1 - R^2_n)$	0.042	1.000	0.281	0.718	0.086
	STCL-CvM	0.042	1.000	0.568	0.973	0.413
	STCL-AD	0.042	1.000	0.595	0.987	0.558

Table 34 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.05$ and $q = 0.50$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.055	0.382	0.078	0.152	0.073
	$A^2_{r,n}$	0.060	0.306	0.059	0.115	0.052
	B^2	0.045	0.089	0.007	0.029	0.010
	STCL-CvM	0.045	0.374	0.093	0.176	0.106
	STCL-AD	0.049	0.294	0.065	0.140	0.102
$n = 40$	$W^2_{r,n}$	0.060	0.782	0.124	0.344	0.111
	$A^2_{r,n}$	0.051	0.718	0.082	0.272	0.096
	B^2	0.051	0.691	0.081	0.293	0.083
	STCL-CvM	0.044	0.737	0.140	0.357	0.192
	STCL-AD	0.048	0.702	0.120	0.334	0.180
$n = 60$	$W^2_{r,n}$	0.062	0.942	0.178	0.534	0.132
	$A^2_{r,n}$	0.062	0.916	0.127	0.461	0.114
	B^2	0.052	0.950	0.165	0.598	0.233
	STCL-CvM	0.041	0.917	0.190	0.559	0.291
	STCL-AD	0.048	0.908	0.175	0.546	0.308
$n = 80$	$W^2_{r,n}$	0.061	0.996	0.244	0.721	0.223
	$A^2_{r,n}$	0.051	0.991	0.189	0.652	0.187
	B^2	0.046	0.990	0.319	0.832	0.427
	STCL-CvM	0.042	0.985	0.269	0.714	0.427
	STCL-AD	0.046	0.985	0.257	0.720	0.487
$n = 100$	$W^2_{r,n}$	0.045	1.000	0.335	0.840	0.279
	$A^2_{r,n}$	0.046	0.998	0.256	0.788	0.236
	B^2	0.056	1.000	0.443	0.939	0.631
	STCL-CvM	0.041	0.997	0.364	0.839	0.557
	STCL-AD	0.045	0.998	0.362	0.862	0.625

Table 35 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.05$ and $q = 0.80$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.063	0.110	0.050	0.084	0.054
	$A^2_{r,n}$	0.063	0.130	0.052	0.085	0.060
	B^2	0.045	0.000	0.009	0.000	0.000
	STCL-CvM	0.045	0.103	0.043	0.080	0.075
	STCL-AD	0.056	0.064	0.040	0.051	0.055
$n = 40$	$W^2_{r,n}$	0.050	0.208	0.063	0.086	0.046
	$A^2_{r,n}$	0.048	0.255	0.067	0.107	0.063
	B^2	0.046	0.003	0.002	0.005	0.000
	STCL-CvM	0.039	0.211	0.078	0.134	0.135
	STCL-AD	0.049	0.166	0.051	0.099	0.100
$n = 60$	$W^2_{r,n}$	0.050	0.323	0.057	0.134	0.060
	$A^2_{r,n}$	0.053	0.400	0.065	0.165	0.076
	B^2	0.056	0.028	0.006	0.016	0.010
	STCL-CvM	0.041	0.335	0.098	0.197	0.235
	STCL-AD	0.044	0.283	0.079	0.162	0.187
$n = 80$	$W^2_{r,n}$	0.044	0.424	0.078	0.181	0.079
	$A^2_{r,n}$	0.045	0.506	0.086	0.211	0.108
	B^2	0.054	0.110	0.017	0.068	0.062
	STCL-CvM	0.053	0.486	0.122	0.285	0.332
	STCL-AD	0.052	0.439	0.108	0.256	0.306
$n = 100$	$W^2_{r,n}$	0.046	0.549	0.077	0.240	0.075
	$A^2_{r,n}$	0.049	0.631	0.081	0.290	0.108
	B^2	0.052	0.271	0.038	0.163	0.182
	STCL-CvM	0.059	0.587	0.131	0.372	0.436
	STCL-AD	0.052	0.554	0.110	0.338	0.410

Table 36 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.025$ and $q = 0.20$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.027	0.697	0.095	0.268	0.073
	$A^2_{r,n}$	0.028	0.584	0.047	0.172	0.052
	B^2	0.033	0.211	0.007	0.035	0.032
	STCL-CvM	0.013	0.490	0.039	0.140	0.038
	STCL-AD	0.021	0.434	0.039	0.112	0.038
$n = 40$	$W^2_{r,n}$	0.020	0.978	0.209	0.597	0.101
	$A^2_{r,n}$	0.017	0.965	0.147	0.546	0.111
	B^2	0.013	0.896	0.060	0.321	0.061
	STCL-CvM	0.009	0.932	0.114	0.429	0.071
	STCL-AD	0.010	0.929	0.098	0.420	0.082
$n = 60$	$W^2_{r,n}$	0.023	0.999	0.301	0.812	0.159
	$A^2_{r,n}$	0.026	0.998	0.276	0.807	0.200
	B^2	0.024	0.995	0.150	0.655	0.119
	STCL-CvM	0.020	0.992	0.191	0.674	0.111
	STCL-AD	0.020	0.996	0.180	0.685	0.140
$n = 80$	$W^2_{r,n}$	0.021	1.000	0.440	0.940	0.240
	$A^2_{r,n}$	0.026	1.000	0.425	0.941	0.314
	B^2	0.021	1.000	0.294	0.867	0.188
	STCL-CvM	0.023	0.999	0.292	0.851	0.146
	STCL-AD	0.025	1.000	0.304	0.873	0.218
$n = 100$	$W^2_{r,n}$	0.026	1.000	0.561	0.984	0.314
	$A^2_{r,n}$	0.019	1.000	0.570	0.990	0.432
	B^2	0.024	1.000	0.436	0.956	0.272
	STCL-CvM	0.018	1.000	0.389	0.930	0.242
	STCL-AD	0.022	1.000	0.429	0.953	0.362

Table 37 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.025$ and $q = 0.50$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.030	0.233	0.040	0.079	0.029
	$A^2_{r,n}$	0.030	0.125	0.016	0.043	0.019
	B^2	0.023	0.021	0.000	0.008	0.003
	STCL-CvM	0.016	0.184	0.036	0.079	0.051
	STCL-AD	0.026	0.134	0.022	0.052	0.039
$n = 40$	$W^2_{r,n}$	0.025	0.660	0.060	0.207	0.056
	$A^2_{r,n}$	0.020	0.493	0.027	0.111	0.034
	B^2	0.028	0.535	0.030	0.153	0.029
	STCL-CvM	0.019	0.572	0.067	0.234	0.095
	STCL-AD	0.021	0.536	0.043	0.196	0.079
$n = 60$	$W^2_{r,n}$	0.033	0.887	0.098	0.375	0.065
	$A^2_{r,n}$	0.028	0.802	0.035	0.247	0.040
	B^2	0.025	0.881	0.085	0.446	0.118
	STCL-CvM	0.019	0.828	0.098	0.366	0.162
	STCL-AD	0.018	0.812	0.083	0.360	0.159
$n = 80$	$W^2_{r,n}$	0.032	0.983	0.153	0.553	0.112
	$A^2_{r,n}$	0.022	0.947	0.082	0.382	0.069
	B^2	0.027	0.981	0.191	0.702	0.286
	STCL-CvM	0.016	0.961	0.156	0.543	0.262
	STCL-AD	0.021	0.959	0.151	0.539	0.290
$n = 100$	$W^2_{r,n}$	0.024	0.997	0.223	0.710	0.159
	$A^2_{r,n}$	0.023	0.990	0.108	0.550	0.098
	B^2	0.023	0.998	0.298	0.867	0.437
	STCL-CvM	0.017	0.987	0.213	0.726	0.369
	STCL-AD	0.015	0.991	0.200	0.742	0.437

Table 38 Empirical Power of Goodness-of-Fit Statistics at $\alpha = 0.025$ and $q = 0.80$. Exponential Distribution with Exponential Censoring

		Alternative Distribution				
		Exponential	Weibull shape=2	Gamma shape=1.5	Gamma shape=2	Lognormal from N(0,1)
$n = 20$	$W^2_{r,n}$	0.034	0.040	0.018	0.025	0.020
	$A^2_{r,n}$	0.033	0.058	0.019	0.031	0.026
	B^2	0.020	0.000	0.007	0.000	0.000
	STCL-CvM	0.015	0.039	0.014	0.022	0.019
	STCL-AD	0.015	0.019	0.012	0.012	0.011
$n = 40$	$W^2_{r,n}$	0.017	0.097	0.028	0.034	0.025
	$A^2_{r,n}$	0.020	0.124	0.028	0.041	0.031
	B^2	0.022	0.000	0.001	0.000	0.000
	STCL-CvM	0.012	0.102	0.028	0.048	0.047
	STCL-AD	0.015	0.060	0.018	0.034	0.028
$n = 60$	$W^2_{r,n}$	0.023	0.174	0.022	0.059	0.026
	$A^2_{r,n}$	0.022	0.210	0.021	0.066	0.031
	B^2	0.027	0.001	0.000	0.002	0.001
	STCL-CvM	0.013	0.185	0.034	0.104	0.113
	STCL-AD	0.020	0.147	0.025	0.072	0.087
$n = 80$	$W^2_{r,n}$	0.021	0.245	0.037	0.084	0.036
	$A^2_{r,n}$	0.024	0.284	0.040	0.092	0.043
	B^2	0.028	0.018	0.000	0.008	0.007
	STCL-CvM	0.021	0.318	0.055	0.156	0.174
	STCL-AD	0.020	0.259	0.047	0.120	0.149
$n = 100$	$W^2_{r,n}$	0.023	0.359	0.029	0.126	0.027
	$A^2_{r,n}$	0.022	0.410	0.030	0.145	0.030
	B^2	0.024	0.085	0.002	0.035	0.029
	STCL-CvM	0.024	0.418	0.059	0.220	0.269
	STCL-AD	0.024	0.373	0.041	0.196	0.238

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

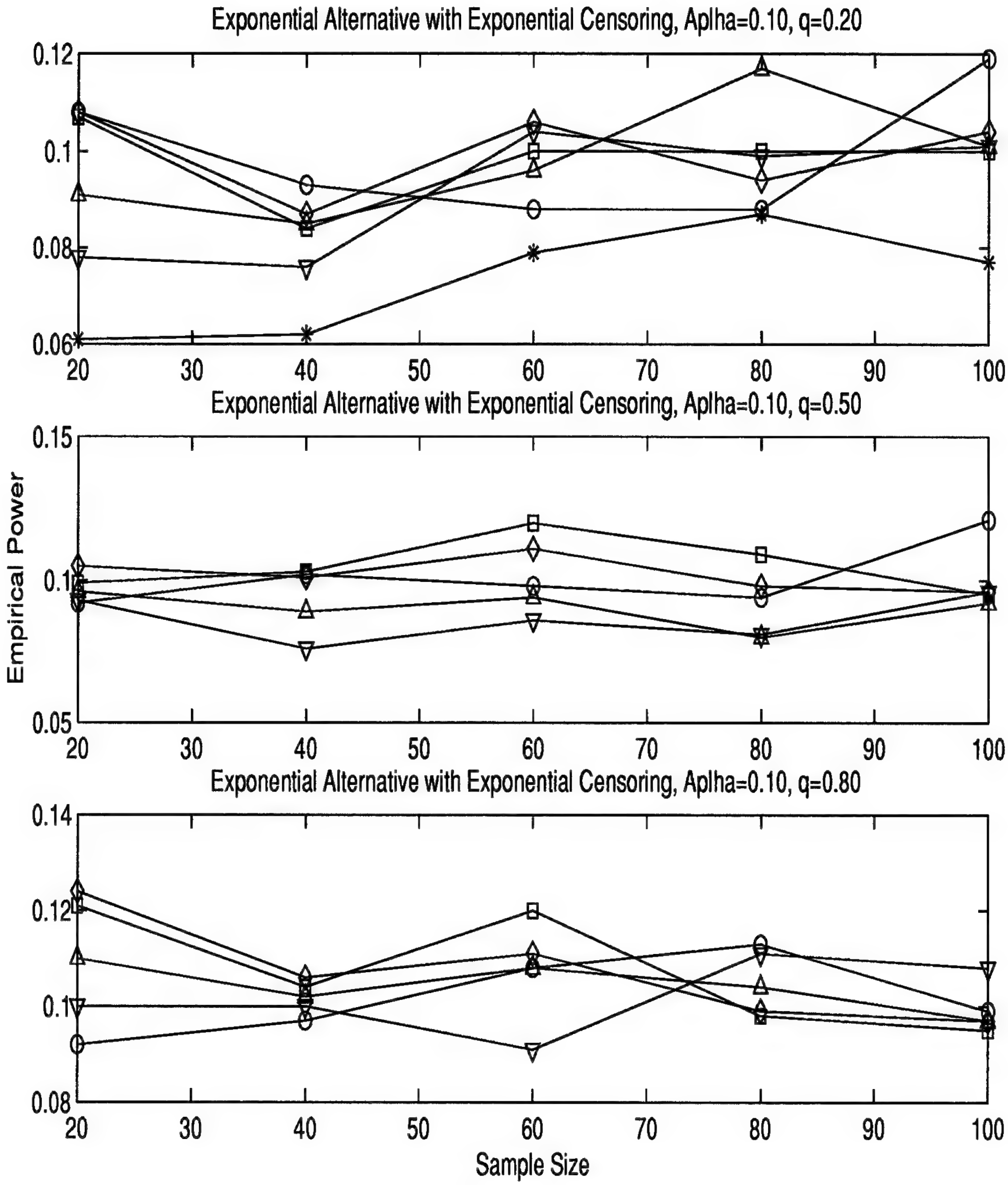

Figure 43 Power Comparison of Tests for Exponentiality, Underlying Distribution is Exponential, $\alpha = 0.10$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	$\square$	STCL-CvM	∇	B^2	O
$A^2_{r,n}$	$\Diamond$	STCL-AD	$\triangle$	$n(1-R^2_n)$	$*$

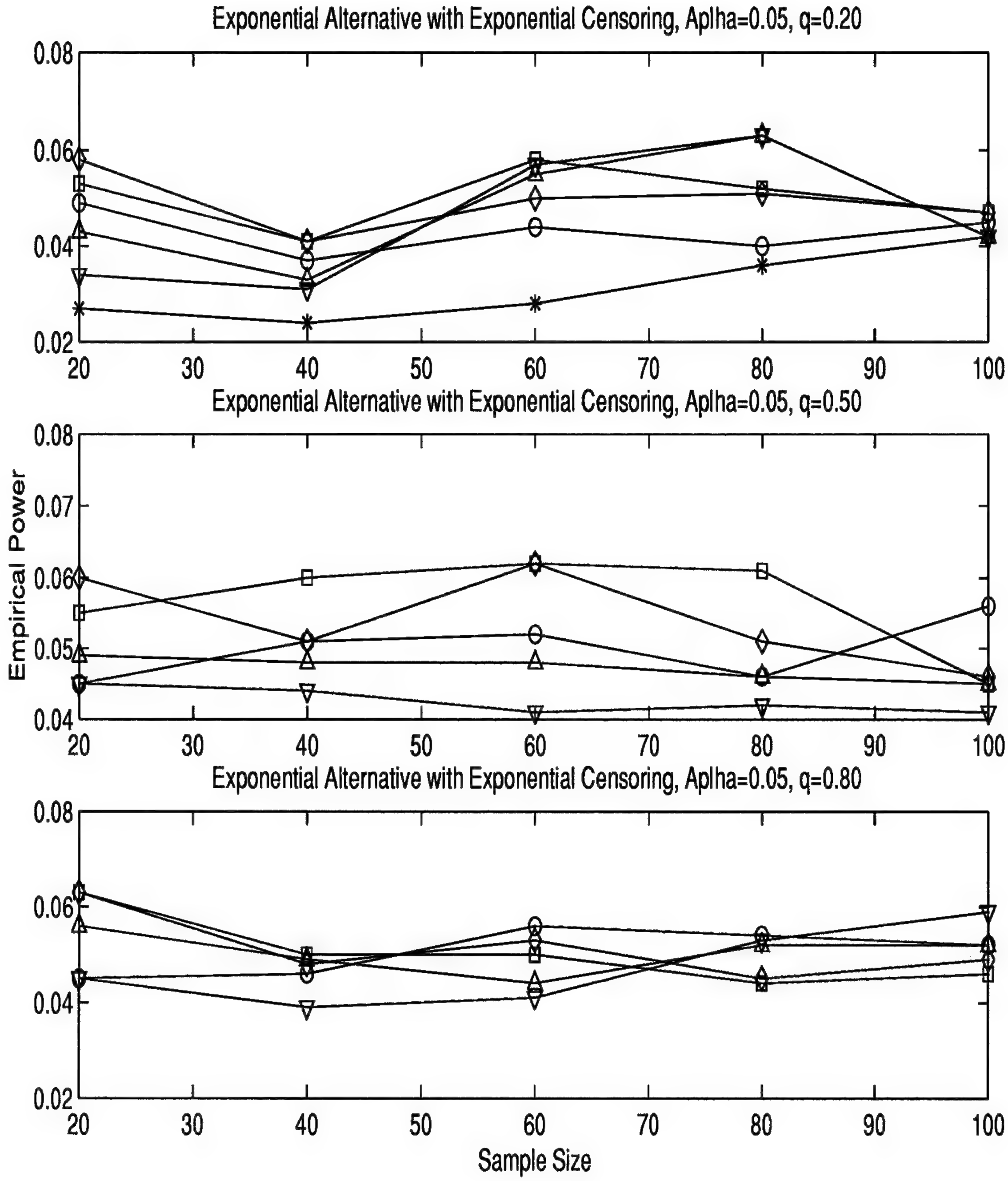

Figure 44 Power Comparison of Tests for Exponentiality, Underlying Distribution is Exponential, $\alpha = 0.05$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

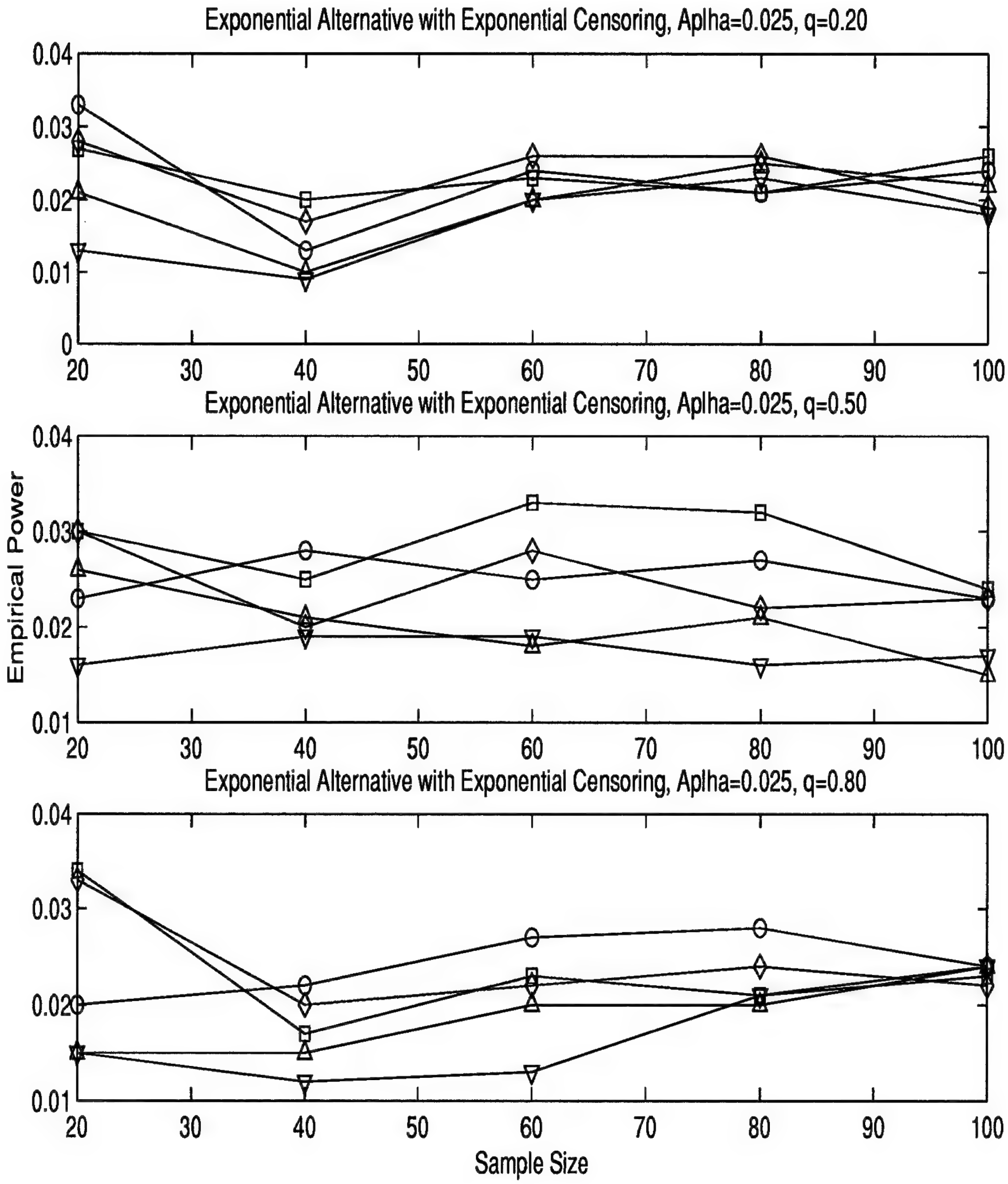

Figure 45 Power Comparison of Tests for Exponentiality, Underlying Distribution is Exponential, $\alpha = 0.025$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◊	STCL-AD	△	$n(1-R^2_n)$	*

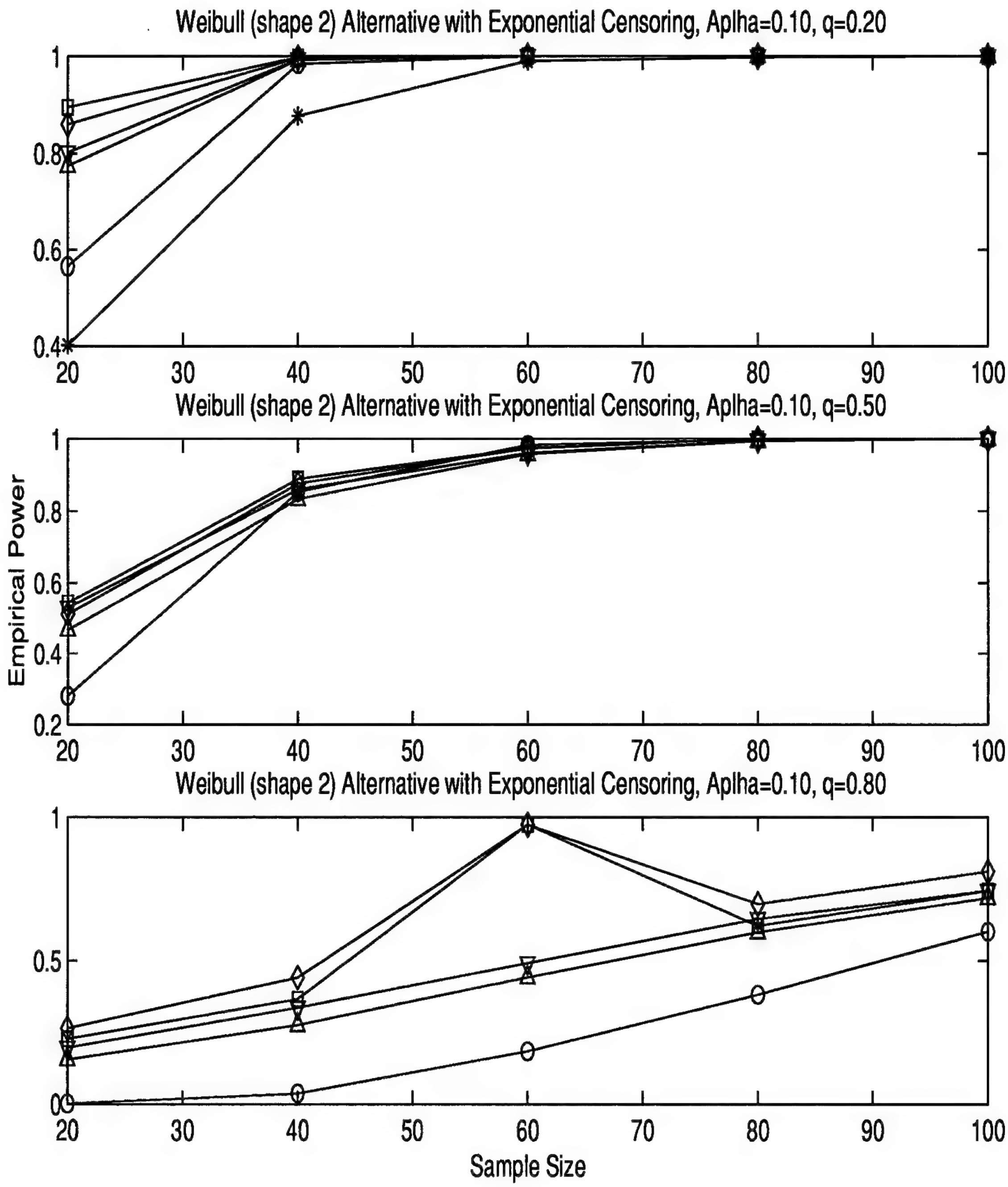

Figure 46 Power Comparison of Tests for Exponentiality, Underlying Distribution is Weibull (shape=2),, $\alpha = 0.10$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

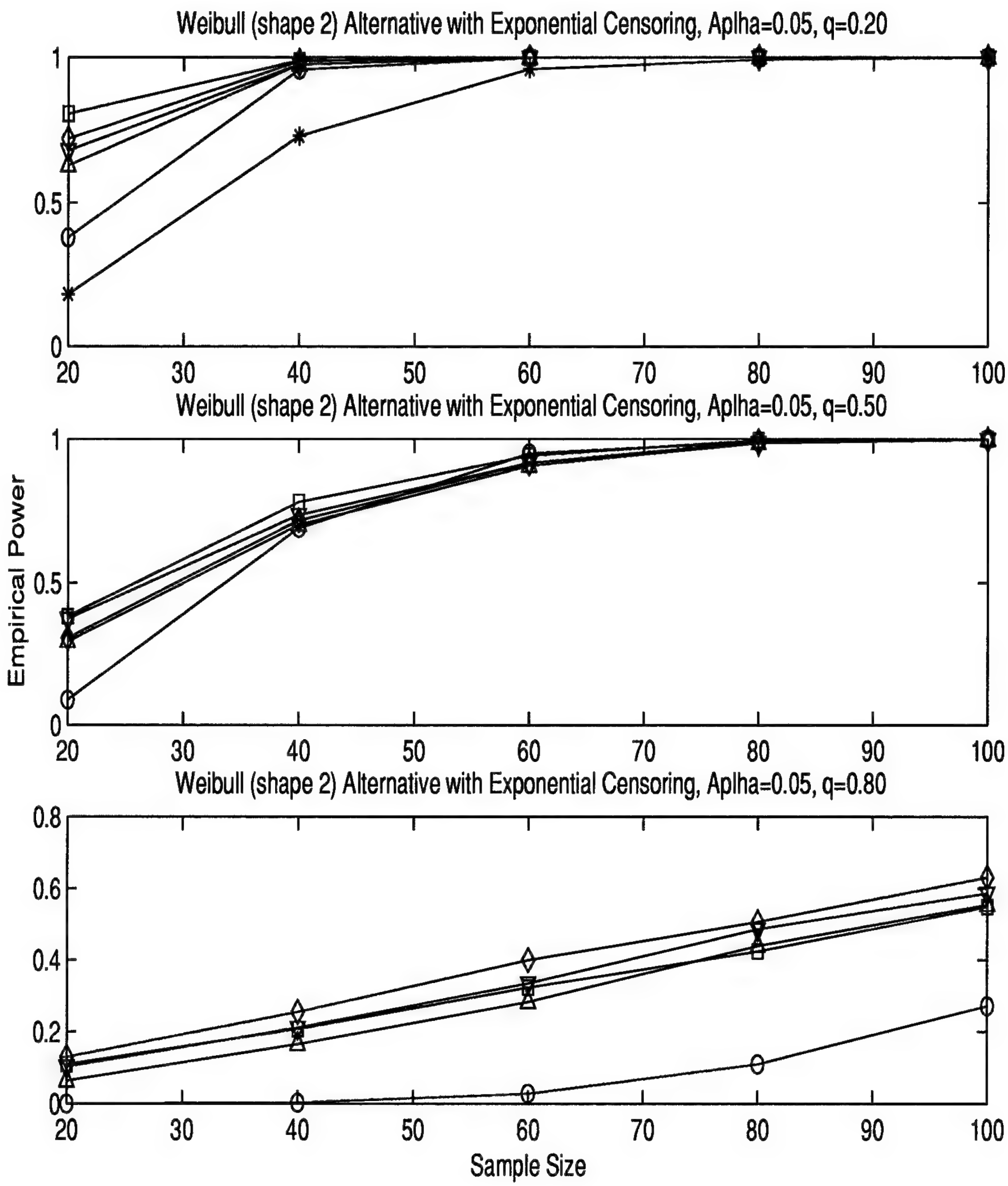

Figure 47 Power Comparison of Tests for Exponentiality, Underlying Distribution is Weibull (shape=2),, $\alpha = 0.05$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W_{r,n}^2$	$\square$	STCL-CvM	∇	B^2	O
$A_{r,n}^2$	$\diamond$	STCL-AD	$\triangle$	$n(1-R_n^2)$	$*$

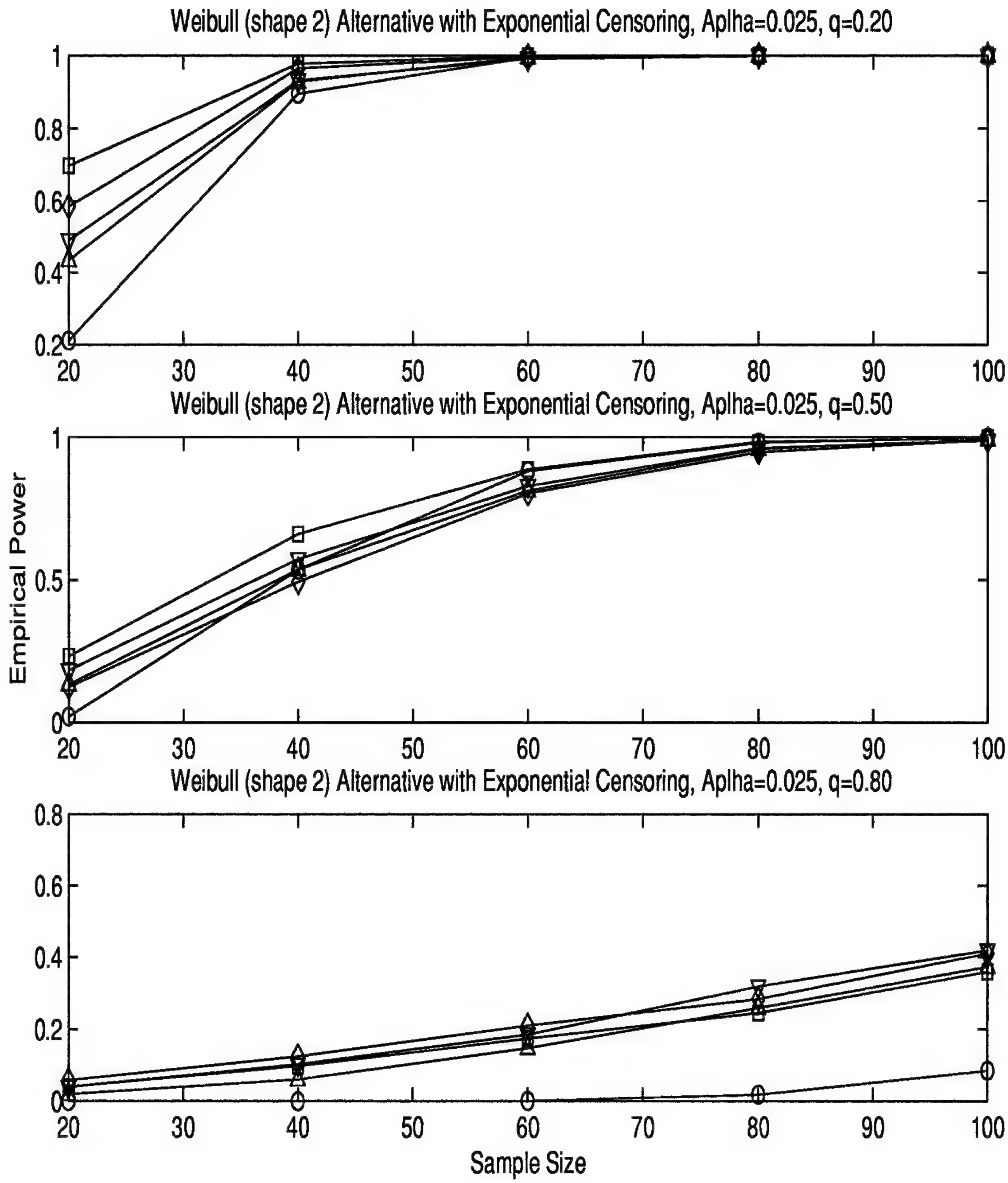

Figure 48 Power Comparison of Tests for Exponentiality, Underlying Distribution is Weibull (shape=2), $\alpha = 0.025$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W_{r,n}^2$	□	STCL-CvM	▽	B^2	O
$A_{r,n}^2$	◇	STCL-AD	△	$n(1-R_n^2)$	*

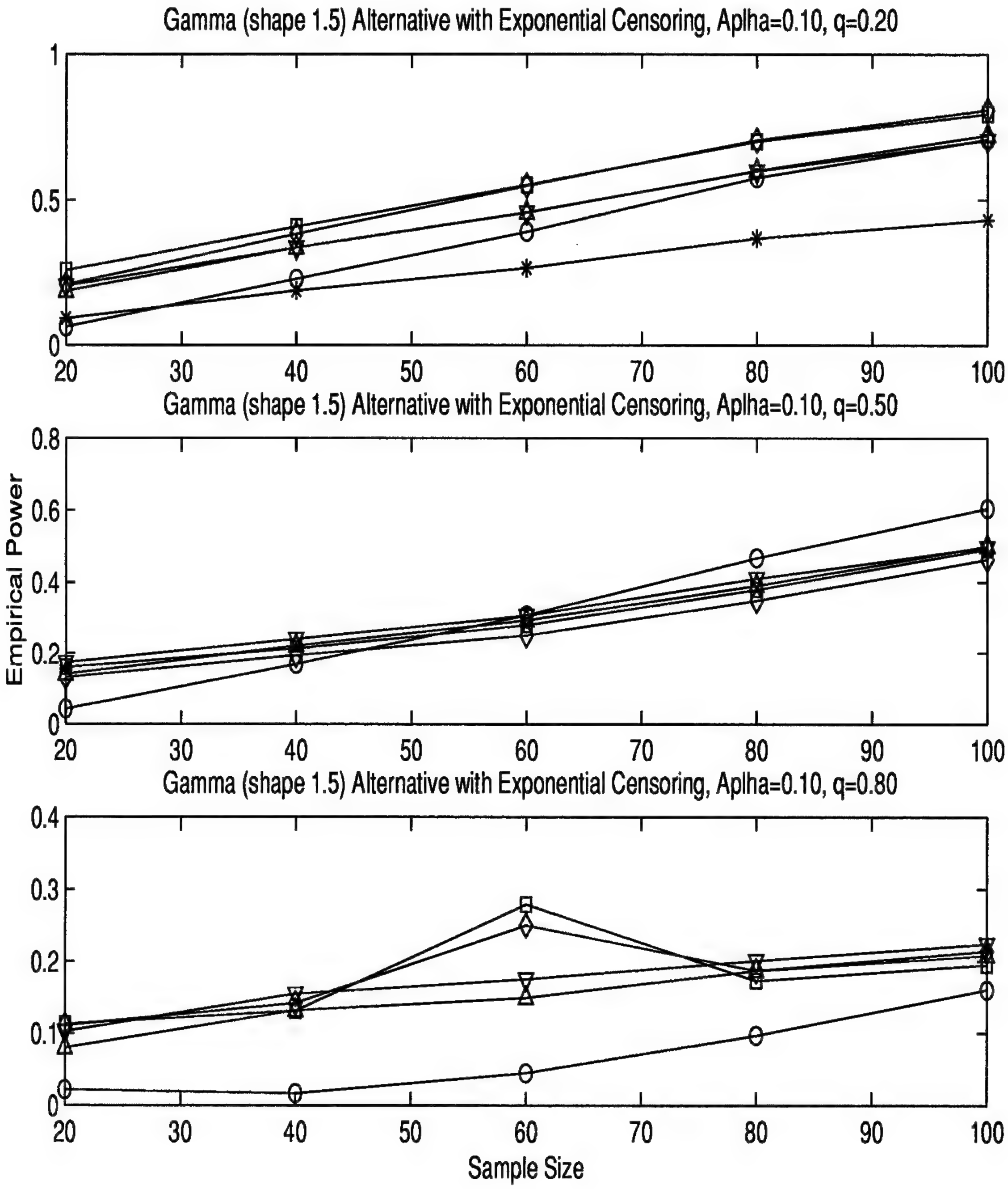

Figure 49 Power Comparison of Tests for Exponentiality, Underlying Distribution is Gamma (shape=1.5), $\alpha = 0.10$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

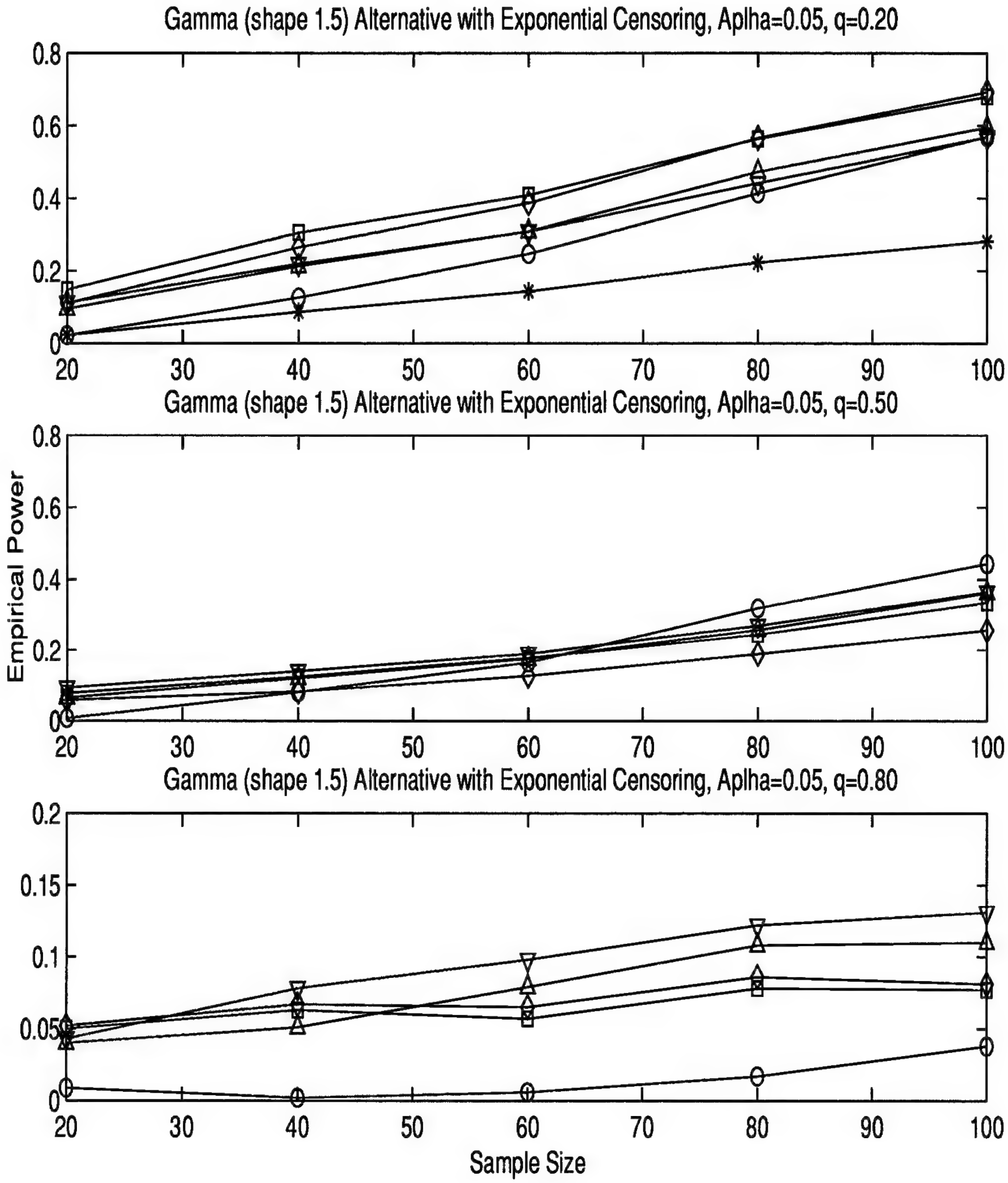

Figure 50 Power Comparison of Tests for Exponentiality, Underlying Distribution is Gamma (shape=1.5), $\alpha = 0.05$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	$\square$	STCL-CvM	∇	B^2	O
$A^2_{r,n}$	$\diamond$	STCL-AD	$\triangle$	$n(1-R^2_n)$	$*$

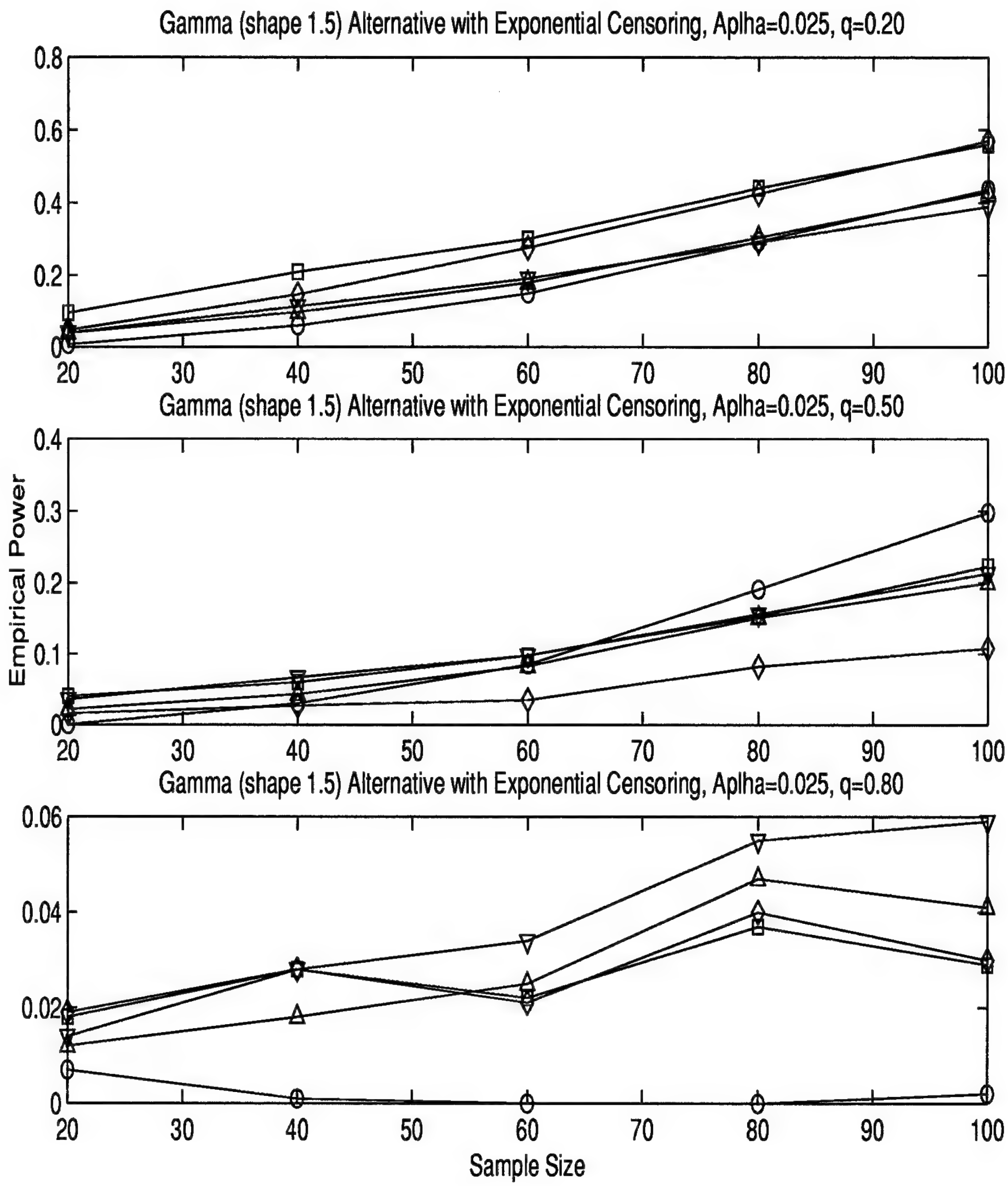

Figure 51 Power Comparison of Tests for Exponentiality, Underlying Distribution is Gamma (shape=1.5), $\alpha = 0.025$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	$\square$	STCL-CvM	$\bigtriangledown$	B^2	O
$A^2_{r,n}$	$\Diamond$	STCL-AD	$\bigtriangleup$	$n(1-R^2_n)$	$*$

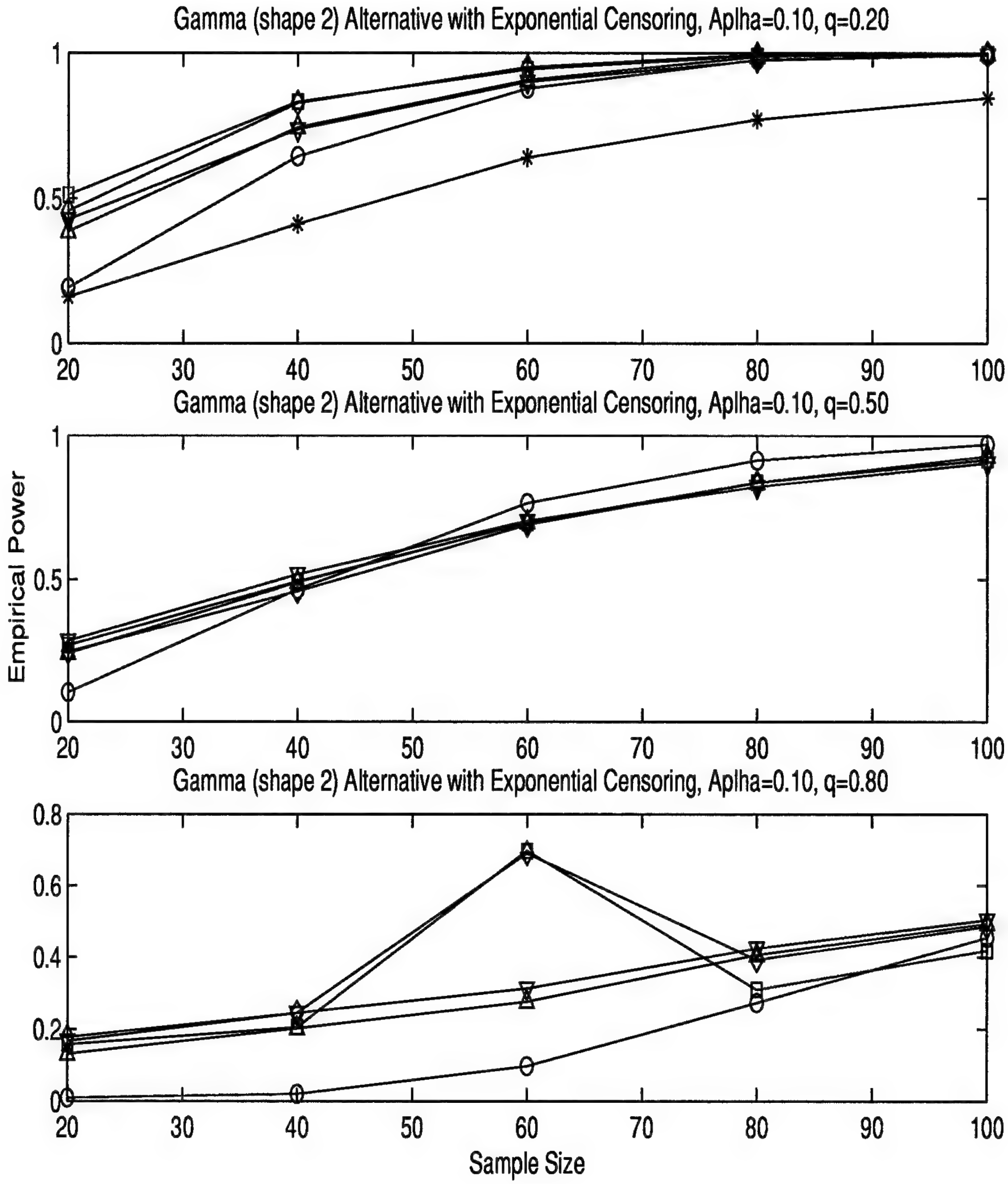

Figure 52 Power Comparison of Tests for Exponentiality, Underlying Distribution is Gamma (shape=2), $\alpha = 0.10$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

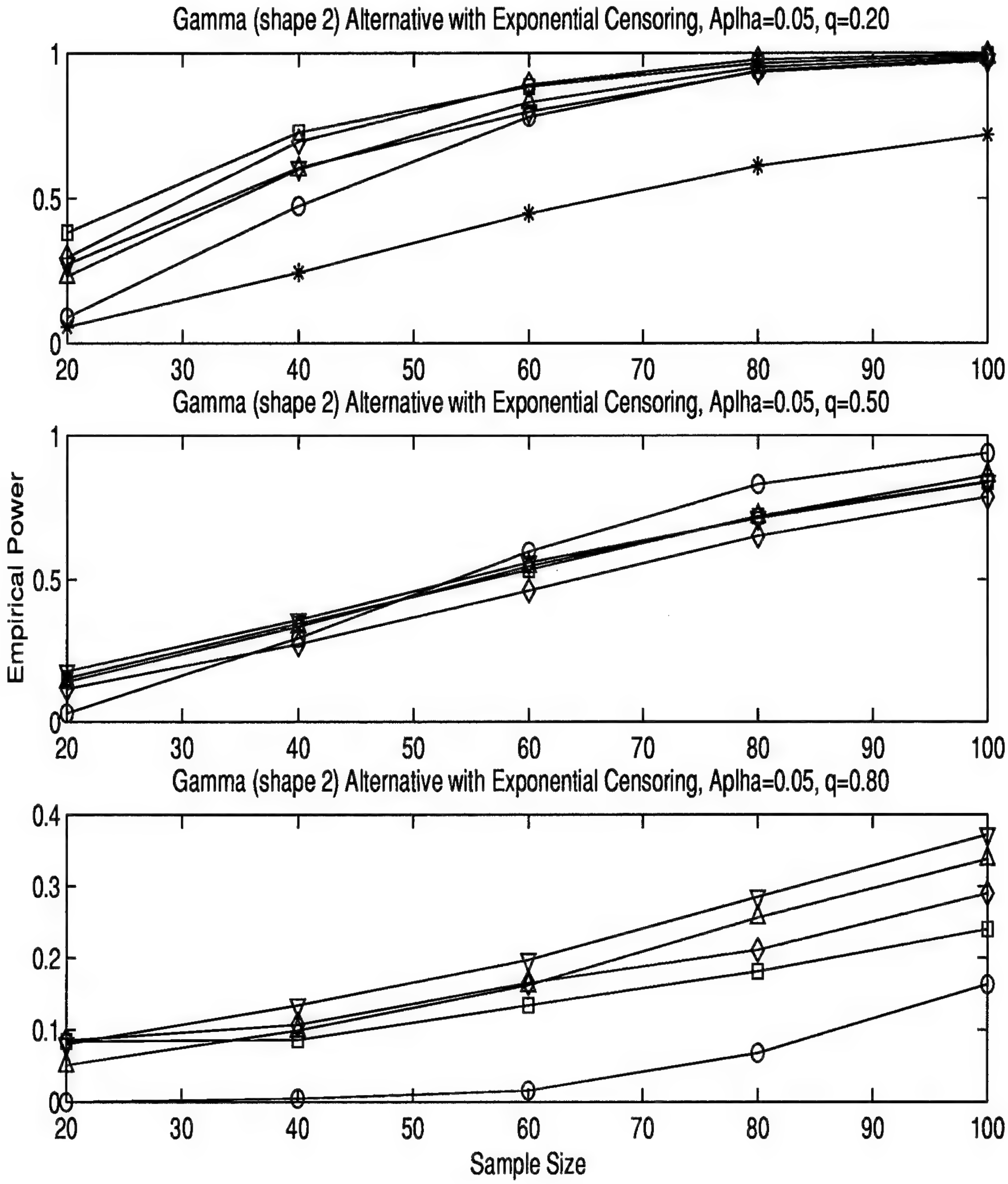

Figure 53 Power Comparison of Tests for Exponentiality, Underlying Distribution is Gamma (shape=2), $\alpha = 0.05$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	$\square$	STCL-CvM	∇	B^2	O
$A^2_{r,n}$	$\diamond$	STCL-AD	$\triangle$	$n(1-R^2_n)$	*

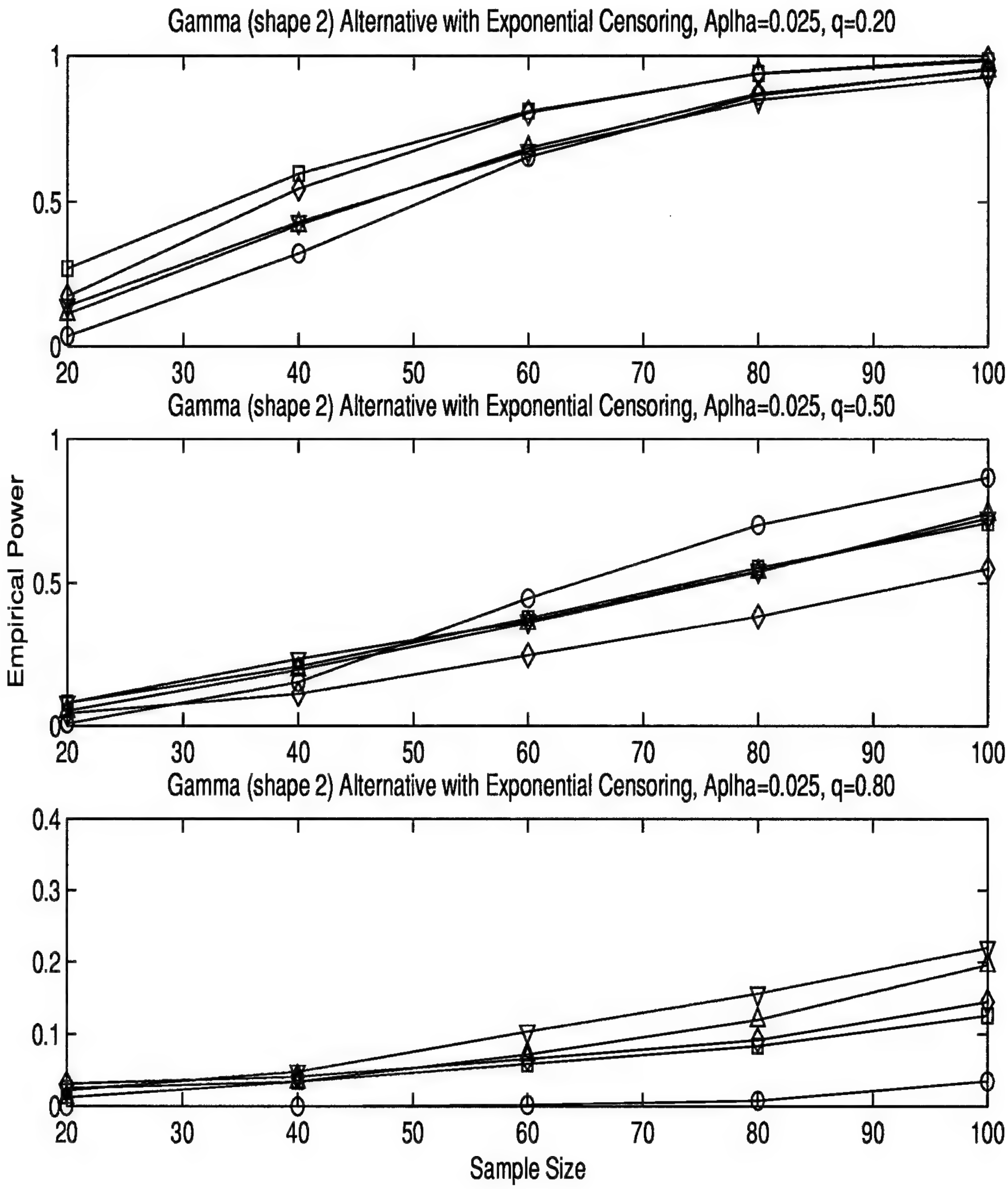

Figure 54 Power Comparison of Tests for Exponentiality, Underlying Distribution is Gamma (shape=2), $\alpha = 0.025$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

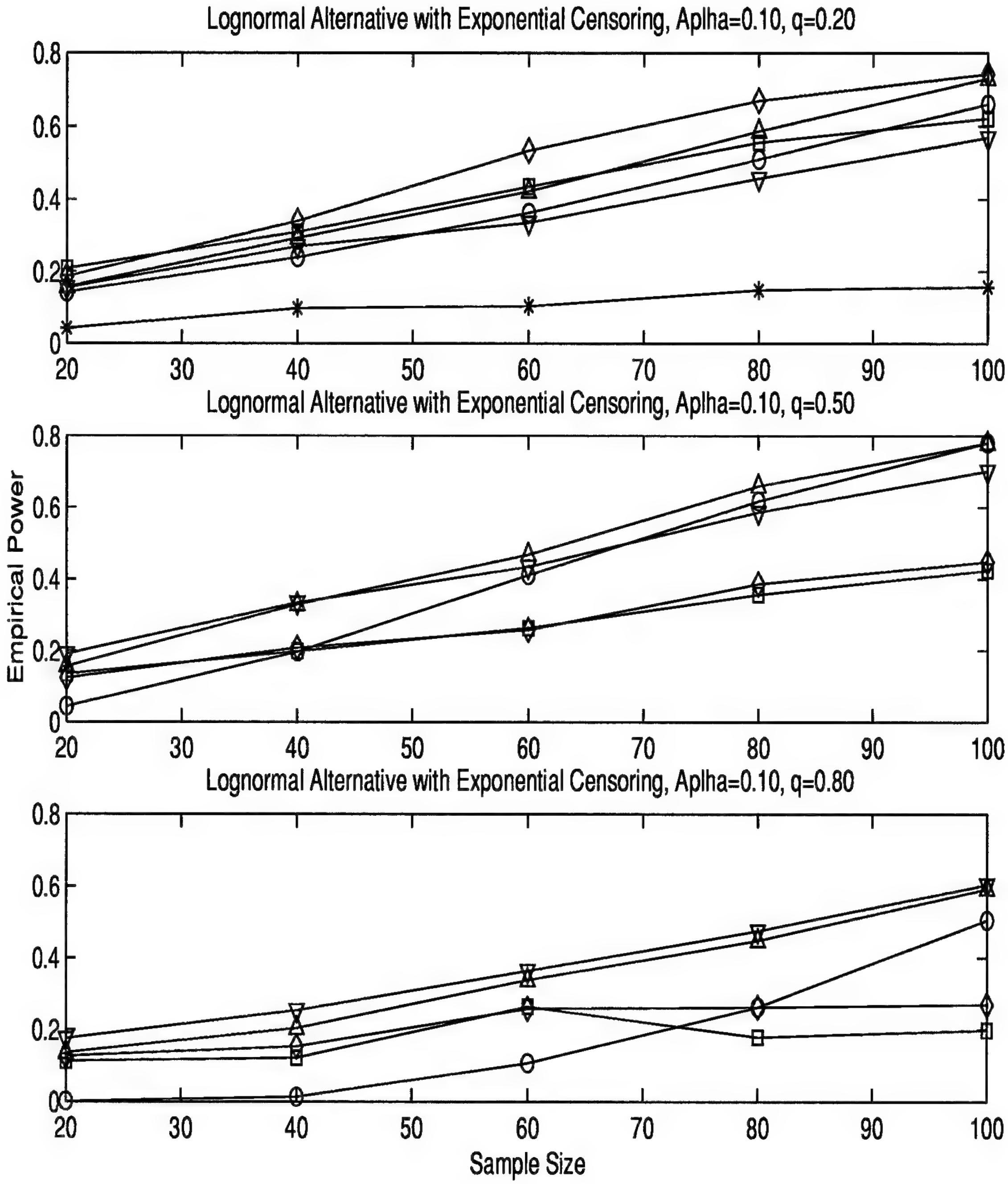

Figure 55 Power Comparison of Tests for Exponentiality, Underlying Distribution is Lognormal from N(0,1), $\alpha = 0.10$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

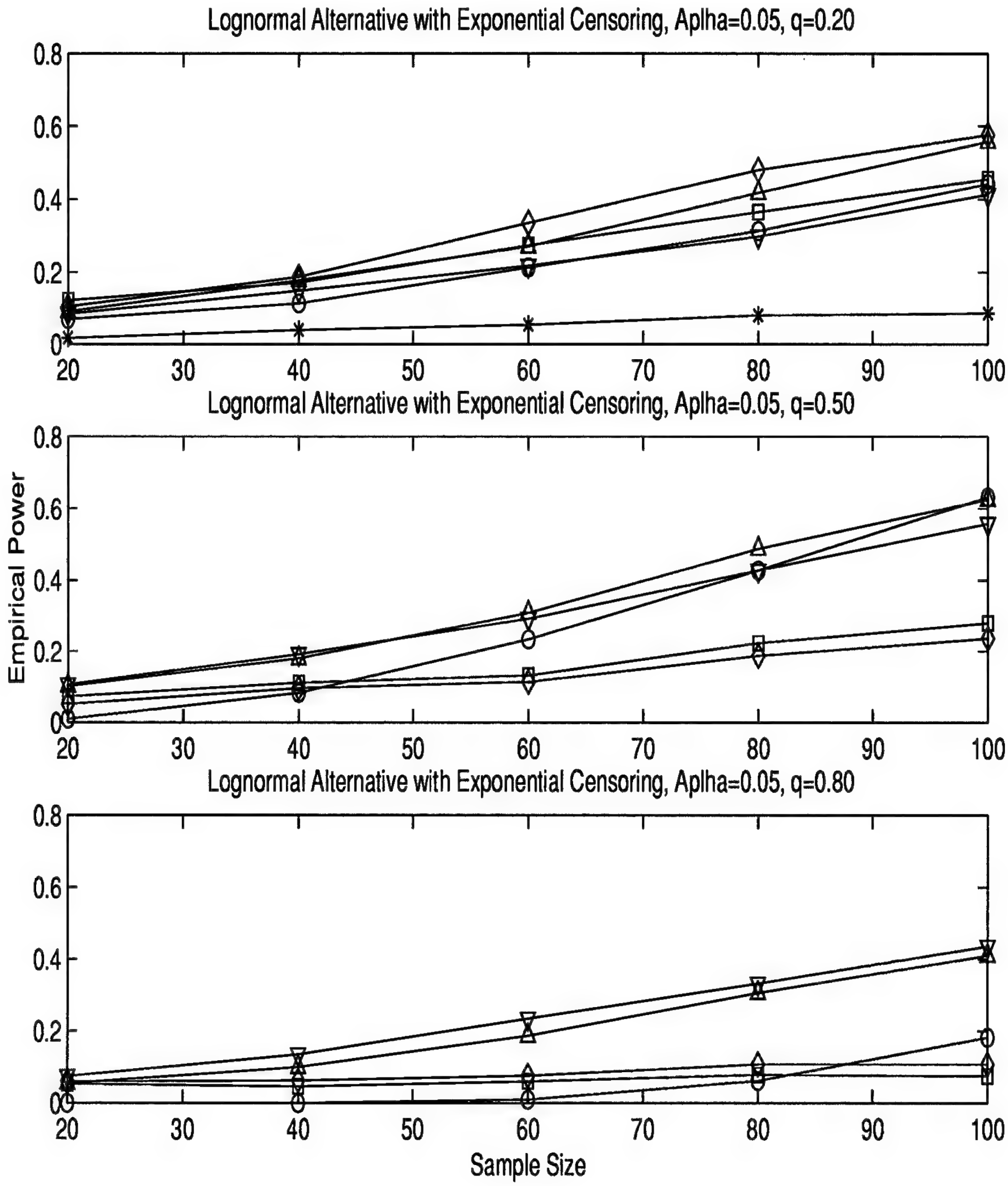

Figure 56 Power Comparison of Tests for Exponentiality, Underlying Distribution is Lognormal from N(0,1), $\alpha = 0.05$.

Legend for Empirical Power Study Plots for Tests for Exponentiality

Test Statistic	Symbol	Test Statistic	Symbol	Test Statistic	Symbol
$W^2_{r,n}$	□	STCL-CvM	▽	B^2	O
$A^2_{r,n}$	◇	STCL-AD	△	$n(1-R^2_n)$	*

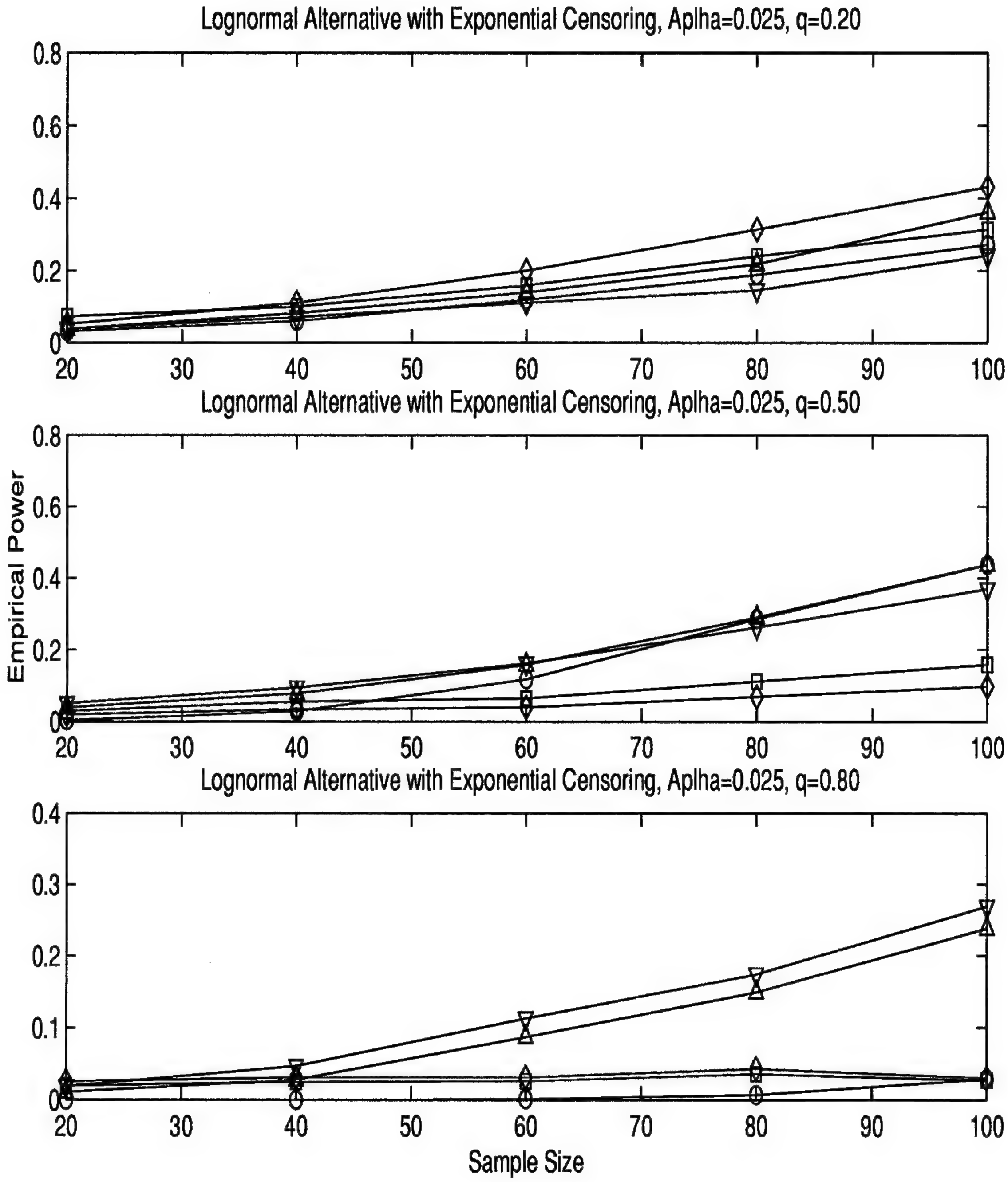

Figure 57 Power Comparison of Tests for Exponentiality, Underlying Distribution is Lognormal from N(0,1), $\alpha = 0.025$.

```
% mle.m

% Maximum Likelihood Estimation for 2-parameter Weibull
% using iterative Newton-Raphson approach given in the Leemis textbook
% "SORTDATA" is a complete set of failure and withdrawal times

%  Written by Dave Reineke, 1997

clear NREM LX LY Lx Bml INITB fail

n=length(SORTDATA);

k=0;
for j=1:n
  if SORTDATA(j,2)==1
    k=k+1;
    fail(k)=SORTDATA(j,1);
  end;
end;

if k >= 2

k=0;
for j=1:n
  if SORTDATA(j,2)==1
    k=k+1;
    NREM(k)=n-j+1;
  end;
end;

LX=log(fail);
LY=log(log((n+1)./NREM));

LXbar=mean(LX);
LYbar=mean(LY);

Nsum=sum((LX-LXbar).*(LY-LYbar));
Dsum=sum((LX-LXbar).^2);

INITB=Nsum/Dsum;

Lx=log(SORTDATA(:,1));

Lsum=sum((SORTDATA(:,1).^INITB).*Lx);
Psum=sum(SORTDATA(:,1).^INITB);
LPsum=sum((Lx.^2).*SORTDATA(:,1).^INITB);

Bml=INITB-(k/INITB+sum(LX)-k*Lsum/Psum)/(-k/INITB^2-k/Psum^2*(Psum*LPsum-Lsum^2));

DIFF=1;
```

```
C=1;
while DIFF > .00001
  C=C+1;
  if C == 1001
    C
    BETAhat=Bml(C);
    break;
  end;

  Lsum=sum((SORTDATA(:,1).^Bml(C-1)).*Lx);
  Psum=sum(SORTDATA(:,1).^Bml(C-1));
  LPsum=sum((Lx.^2).*SORTDATA(:,1).^Bml(C-1));

  Bml(C)=Bml(C-1)-(k/Bml(C-1)+sum(LX)-k*Lsum/Psum)/(-k/Bml(C-1)^2-k/Psum^2*(Psum*LPsum-Lsum^2));

  DIFF=abs(Bml(C) - Bml(C-1));

end;
end;

BETAhat=Bml(C);

ETAhat=(sum(SORTDATA(:,1).^BETAhat)/k).^(1/BETAhat);

MUhat=ETAhat*gamma(1/BETAhat + 1);
```

Appendix H. Matlab Code for Minimum Distance Estimation of a Weibull Location Parameter

```
% mde.m Finds the minimum distance estimate (MDE) of the location
%    parameter of a 3 parameter Weibull distribution.

% Written by Dave Reineke, 1998

% SORTDATA is a n by 2 matrix of failure and withdrawal times ( - loc).
%       FAIL is the set of failure times only.
% KME is the Kaplan-Meier estimator of the distribution function.


t=1-exp(-((FAIL-GAMhat1)./ETAhat1).^BETAhat1);

% MDE using the Cramer-von Mises statistic (KME vs MLE)

diff=1;

%  Golden Search algorithm

alf=2/(1+sqrt(5));

lt=0;
rt=min(data(:,1));

c1=0;
while diff > .00000001
  c1=c1+1;
  if c1 == 1000
    c1
    GAMhatW=min(data(:,1));
    break;
  end;
  x1=lt + (1-alf)*(rt-lt);
  x2=lt + alf*(rt-lt);

  t=1-exp(-((FAIL-x1)./ETAhat1).^BETAhat1);
  W1md=0;
  for i=1:r-1
    W1md=W1md + (KME(i)^2)*(t(i+1)-t(i)) - KME(i)*(t(i+1)^2-t(i)^2);
  end;
  W2md1=n/3 + n*(W1md + t(r)^2-t(r));

  t=1-exp(-((FAIL-x2)./ETAhat1).^BETAhat1);
  w2md=0;
  for i=1:r-1
    w2md=w2md + (KME(i)^2)*(t(i+1)-t(i)) - KME(i)*(t(i+1)^2-t(i)^2);
  end;
  W2md2=n/3 + n*(w2md + t(r)^2-t(r));

  if W2md1 < W2md2
    rt=x2;
```

```
    GAMhatW=x1;
  else
    lt=x1;
    GAMhatW=x2;
  end;

  diff=abs(W2md1-W2md2);

end;

% ***************************************************************************
% MDE using the Anderson-Darling statistic (KME vs MLE)

diff=1;

%  Golden Search algorithm

alf=2/(1+sqrt(5));

lt=0;
rt=.999*min(data(:,1));

c2=0;
while diff > .00000001
  c2=c2+1;
  if c2 == 1000
    c2
    GAMhatA=min(data(:,1));
    break;
  end;
  x1=lt + (1-alf)*(rt-lt);
  x2=lt + alf*(rt-lt);

  t=1-exp(-((FAIL-x1)./ETAhat1).^BETAhat1);

  [tmin1,I1]=min(t);
  [tmax1,I3]=max(t);

  if tmin1==0
    if tmax1 == 1
      if I3-1-I1==1

        A2md1=-n+n*(-log(t(I1+1)) - log(1-t(I1+1)));

      elseif I3-1-I1>1

        A1md=-(KME(I1+1)^2)*log(t(I1+1))-(1-(KME(I1+1)-1)^2)*log(1-t(I1+1));
        for i=I1+1:I3-1
          A1md=A1md +   (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
        end;
        A2md1=-n+n*A1md;
```

```
    end;

  elseif tmax1 < 1
    if r-I1==1

      A2md1=-n+n*(-log(t(r)) - log(1-t(r)));

    elseif r-I1>1

      A1md=-(KME(I1+1)^2)*log(t(I1+1))-(1-(KME(I1+1)-1)^2)*log(1-t(I1+1));
      for i=I1+2:r
        A1md=A1md +   (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md1=-n+n*A1md;
    end;
  end;

elseif tmin1 > 0
  if tmax1==1
    if I3==2

      A2md1=-n+n*(-log(t(1)) - log(1-t(1)));

    elseif I3 > 2

      A1md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
      for i=2:I3-1
        A1md=A1md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md1=-n+n*A1md;
    end;

  elseif tmax1 < 1

    A1md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
    for i=2:r
      A1md=A1md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
    end;
    A2md1=-n+n*A1md;

  end;
end;

t=1-exp(-((FAIL-x2)./ETAhat1).^BETAhat1);

[tmin2,I2]=min(t);
[tmax2,I4]=max(t);

if tmin2==0
  if tmax2 == 1
    if I4-1-I2==1
```

```
      A2md2=-n+n*(-log(t(I2+1)) - log(1-t(I2+1)));

    elseif I4-1-I2>1

      A2md=-(KME(I2+1)^2)*log(t(I2+1))-(1-(KME(I2+1)-1)^2)*log(1-t(I2+1));
      for i=I2+1:I4-1
        A2md=A2md +   (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md2=-n+n*A2md;
    end;

  elseif tmax2 < 1
    if r-I2==1

      A2md2=-n+n*(-log(t(r)) - log(1-t(r)));

    elseif r-I2>1

      A2md=-(KME(I2+1)^2)*log(t(I2+1))-(1-(KME(I2+1)-1)^2)*log(1-t(I2+1));
      for i=I2+2:r
        A2md=A2md +   (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md2=-n+n*A2md;
    end;
  end;

elseif tmin2 > 0
  if tmax2==1
    if I4==2

      A2md2=-n+n*(-log(t(1)) - log(1-t(1)));

    elseif I4 > 2

      A2md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
      for i=2:I4-1
        A2md=A2md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md2=-n+n*A2md;
    end;

  elseif tmax2 < 1

    A2md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
    for i=2:r
      A2md=A2md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
    end;
    A2md2=-n+n*A2md;

  end;
```

```
  end;

  if A2md1 < A2md2
    rt=x2;
    GAMhatA=x1;
  else
    lt=x1;
    GAMhatA=x2;
  end;

  diff=abs(A2md1-A2md2);

end;
```

```
% MISEexp.m

% This program estimates the Mean Integrated Squared Error(MISE)
% for kernel density estimators, ML estimators, the KME, and the PEXE for
% randomly censored data

% Written by Dave Reineke, 1998


nrep=25000;
for n=20;

%  Numerical integration will be performed using Simpson's rule with
% 101 function evaluations.  The weight vector is as follows:

tic

wt(1)=1;
for i=2:2:100
  wt(i)=4;
end;
for i=3:2:99
  wt(i)=2;
end;
wt(101)=1;

rand('seed',1);

beta=1;
eta=50;
loc=0;
q=.80;
Theta=eta*(1-q)/q;
m=0;

while m < nrep

clear FAIL RATE EXSUM PEXE STTT phi KM KME s H sf P J


% Construct a randomly censored data set

for i=1:n
  T(i)=eta*(-log(1-rand)).^(1/beta) + loc;
  C=exprnd(Theta) + loc;
  X(i,1)=min(T(i),C);
  if C<T(i)
    X(i,2)=0;
  else
    X(i,2)=1;
  end;
```

```
end;

T=sort(T);

[data,I]=sort(X(:,1));
for i=1:n
  data(i,2)=X(I(i),2);
end;

r=0;
for i=1:n
  if data(i,2)==1
    r=r+1;
    FAIL(r)=data(i,1);
  end;
end;

if r>1
m=m+1;
qobs(m)=1-r/n;

% Find the ML estimate of scale

ETA=sum(data(:,1))/r;

lower=0;
upper=max(T);
dx=(upper-lower)/100;
x=lower:dx:upper;

F=1-exp(-x./eta);

Fnmle=(F - (1-exp(-x./ETA))).^2;

Imle(m)=wt*Fnmle'*dx/3;


% Construct the Kaplan-Meier PL estimator of the df

for j=1:n
  KM(j)=((n-j)/(n-j+1))^(data(j,2));
end;
KM=cumprod(KM);

if data(n,2)==0
  KM(n)=0;
end;

for j=1:length(x)

  if x(j)<=data(1,1)
```

```
    KME(j)=1;
  end;

  for i=2:n

    if x(j)>data(i-1,1)
      if x(j)<data(i,1)
        KME(j)=KM(i-1);
      end;
    end;
  end;


  if x(j) >= data(n)
    KME(j)=0;
  end;

end;

FnKME=(F - (1-KME)).^2;

IKME(m)=wt*FnKME'*dx/3;

% Construct Sweeder's estimator using the KME

k=4;
mm=floor(n/k);
R=n-k*mm;

for l=1:R
  for j=1:mm+1
    YX(j,l)=data(l+k*(j-1),1);
    YI(j,l)=data(l+k*(j-1),2);
  end;
end;

for l=R+1:k
  for j=1:mm
    YX(j,l)=data(l+k*(j-1),1);
    YI(j,l)=data(l+k*(j-1),2);
  end;
end;

for l=1:R
  for j=1:mm+1
    kmsub(j,l)=((mm+1-j)/(mm+1-j+1)).^YI(j,l);
  end;
end;


for l=R+1:k
```

```
  for j=1:mm
    kmsub(j,1)=((mm-j)/(mm-j+1)).^YI(j,1);
  end;
end;

KMsub=1-cumprod(kmsub);


for l=1:R

extrap(1,1)=max((2*YX(1,l)-YX(2,l)),data(1,1));
extrap(2,1)=min((2*YX(mm,l)-YX(mm-1,l)),data(n,1));

for j=1:length(x)

    if x(j)< extrap(1,1)
    Fhatsub(j,l)=0;
  end;

  if x(j) >=extrap(1,1)
    if x(j) < YX(1,l)
      Fhatsub(j,l)=((KMsub(1,l))/2)*(1-cos(pi*(x(j)-extrap(1,1))/(YX(1,l)-extrap(1,1))));
    end;
  end;

  for i=1:mm

    if x(j)>=YX(i,l)
      if x(j)< YX(i+1,l)
     Fhatsub(j,l)=KMsub(i,l)+((KMsub(i+1,l)-KMsub(i,l))/2)*(1-cos(pi*(x(j)-YX(i,l))/(YX(i+1,l)-YX(
      end;
    end;
  end;

  if x(j) >= YX(mm+1,l)
    if x(j) < extrap(2,1)
      Fhatsub(j,l)=KMsub(mm+1,l)+((1-KMsub(mm+1,l))/2)*(1-cos(pi*(x(j)-YX(mm+1,l))/(extrap(2,1)-YX
    end;
  end;

  if x(j) >= extrap(2,1)
    Fhatsub(j,l)=1;
  end;
end;
end;

for l=R+1:k

extrap(1,1)=max((2*YX(1,l)-YX(2,l)),data(1,1));
extrap(2,1)=min((2*YX(mm,l)-YX(mm-1,l)),data(n,1));
```

```
for j=1:length(x)

  if x(j)< extrap(1,1)
    Fhatsub(j,1)=0;
  end;

  if x(j) >=extrap(1,1)
    if x(j) < YX(1,1)
      Fhatsub(j,1)=((KMsub(1,1))/2)*(1-cos(pi*(x(j)-extrap(1,1))/(YX(1,1)-extrap(1,1))));
    end;
  end;

  for i=1:mm-1

    if x(j)>=YX(i,1)
      if x(j)< YX(i+1,1)
     Fhatsub(j,1)=KMsub(i,1)+((KMsub(i+1,1)-KMsub(i,1))/2)*(1-cos(pi*(x(j)-YX(i,1))/(YX(i+1,1)-YX(
      end;
    end;
  end;


  if x(j) >= YX(mm,1)
    if x(j) < extrap(2,1)
      Fhatsub(j,1)=KMsub(mm,1)+((1-KMsub(mm,1))/2)*(1-cos(pi*(x(j)-YX(mm,1))/(extrap(2,1)-YX(mm,1)
    end;
  end;

  if x(j) >= extrap(2,1)
    Fhatsub(j,1)=1;
  end;

end;
end;

Fhat=mean(Fhatsub');

Fn=(F-Fhat).^2;
ITSJKME(m)=wt*Fn'*dx/3;

% Mean Order Number Estimator

r=0;

if data(1,2)==1
  r=1;
  P(1)=1;
  J(1)=1;
end;

for i=2:n
```

```
  if data(i,2)==1
    r=r+1;
    if r==1
      P(1)=(n+1)/(n-i+2);
      J(1)=P(1);
    elseif r > 1
      if data(i-1,2)==0
        J(r)=(n+1-P(r-1))/(n-i+2);
      else
        J(r)=J(r-1);
      end;
    P(r)=P(r-1)+J(r);
    end;
  end;
end;

sf=P/n;

for j=1:length(x)

  if x(j)<=FAIL(1)
    MON(j)=0;
  end;

  for i=2:r

    if x(j)>FAIL(i-1)
      if x(j)<=FAIL(i)
        MON(j)=sf(i-1);
      end;
    end;
  end;


  if x(j) > FAIL(r)
    MON(j)=1;
  end;
end;

Fn=(F-MON).^2;
IMON(m)=wt*Fn'*dx/3;

% Construct the PEXE of the df

k=0; N=0; R=0;
if data(1,2)==1, k=1;
  STTT(1)=n*data(1,1);
elseif data(1,2)==0, R=data(1,1);
  N=1;
end;
```

```
for i=2:n
  if data(i,2)==1
    if k==0
      k=k+1;
      STTT(k)=(n-i+1)*data(i,1) + R;
    else
      k=k+1;
      STTT(k)=(n-i+1)*data(i,1) + R - (N+n-i+1)*data(i-(N+1),1);
    end;
    R=0;  N=0;
  elseif data(i,2)==0
    R=R+data(i,1);
    N=N+1;
  end;
end;

for i=1:k
  RATE(i)=1/STTT(i);
end;

NEXT=0;
EXSUM(1)=RATE(1)*FAIL(1);
for i=2:k
  NEXT=RATE(i)*(FAIL(i) - FAIL(i-1));
  EXSUM(i)=EXSUM(i-1) + NEXT;
end;

PTTT(1)=-(1/RATE(1))*(exp(-EXSUM(1)) - 1);
for i=2:k
  PIECE=-(1/RATE(i))*(exp(-EXSUM(i)) - exp(-EXSUM(i-1)));
  PTTT(i)=PTTT(i-1) + PIECE;
end;

for i=1:k
  PCDF(i)=1-exp(-EXSUM(i));
end;

Ehat=FAIL(r)/sqrt(-log(1-PCDF(r)));

for j=1:length(x)

  if x(j)<=FAIL(1)
    PEXE(j)=1-exp(-RATE(1)*x(j));
  end;

  for i=2:r

    if x(j)>FAIL(i-1)
      if x(j)<=FAIL(i)
        PEXE(j)=1-exp(-EXSUM(i-1)-RATE(i)*(x(j)-FAIL(i-1)));
      end;
```

```
    end;
  end;


  if x(j) > FAIL(r)
      PEXE(j)=1-exp(-EXSUM(r)-RATE(r)*(x(j)-FAIL(r)));
  end;
end;

FnPEXE=(F - PEXE).^2;

IPEXE(m)=wt*FnPEXE'*dx/3;


% Construct the Foldes-Rejto-Winter kernel estimate of the df
sd=std(FAIL);
h=sd*n^(-1/5);

s(1)=1-KM(2);
for j=2:n-1
  s(j)=KM(j)-KM(j+1);
end;
s(n)=KM(n);

DAT=data(:,1);
keFRW=0;
for j=1:n
  keFRW=keFRW + s(j)*normcdf((x'-DAT(j,:)*ones(size(x')))/h);
end;

FnKEFRW=(F - keFRW').^2;

IKEFRW(m)=wt*FnKEFRW'*dx/3;


% Construct the Blum-Susarla kernel estimate of the df

for j=1:n
  H(j)=((n-j+1)/(n-j+2))^(1-data(j,2));
end;
H=cumprod(H);

DAT=data(:,1);
keBS=0;
for j=1:n
  keBS=keBS + data(j,2)*normcdf((x'-DAT(j,:)*ones(size(x')))/h)/(H(j)*n);
end;

FnKEBS=(F - keBS').^2;

IKEBS(m)=wt*FnKEBS'*dx/3;
```

```
% Klein, Lee, & Moeschberger Survivor Function Estimator

SORTDATA=data;
mle

for j=1:length(x)
  for i=1:n
    if x(j) < data(i,1)
      phi(i,j)=1;
    else
      if data(i,2)==1
        phi(i,j)=0;
      else
        phi(i,j)=exp(-(x(j)/ETAhat).^BETAhat + (data(i,1)/ETAhat).^BETAhat);
      end;
    end;
  end;
end;

KLM=sum(phi)/n;

FnKLM=(F - (1-KLM)).^2;

IKLM(m)=wt*FnKLM'*dx/3;


end;
end;

qbar=mean(qobs);

mImle=mean(Imle);
sImle=std(Imle);

mIKME=mean(IKME);
sIKME=std(IKME);

mITSJKME=mean(ITSJKME);
sITSJKME=std(ITSJKME);

mIMON=mean(IMON);
sIMON=std(IMON);

mIPEXE=mean(IPEXE);
sIPEXE=std(IPEXE);

mIKEFRW=mean(IKEFRW);
sIKEFRW=std(IKEFRW);
```

```
mIKEBS=mean(IKEBS);
sIKEBS=std(IKEBS);

mIKLM=nanmean(IKLM);
sIKLM=nanstd(IKLM);


fid=fopen('MISEexp8.txt','a');
fprintf(fid,'Output for samples of size %2.0f \n',n);
fprintf(fid,'  (Monte Carlo size = %g) \n',nrep);
fprintf(fid,'Expected proportion of censoring  q = %1.2f\n',q);
fprintf(fid,'Observed proportion of censoring  q = %1.2f\n',qbar);
fprintf(fid,'   \n');
fprintf(fid,'Failure distn: Exp. scale=%g\n',eta);
fprintf(fid,'   \n');
fprintf(fid,'Censoring distn: Exponential  scale=%g\n',Theta);
fprintf(fid,'   \n');
fprintf(fid,'*************************************************************\n');
fprintf(fid,'   \n');
fprintf(fid,'MISE for the following df estimators\n');
fprintf(fid,'   \n');
fprintf(fid,'MLE  KME  TSJKME  MONE  PEXE  FRWKE  BSKE  KLME\n');
fprintf(fid,'mean:\n');
fprintf(fid,'%1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f\n',mImle,mIKME,mITSJKME
fprintf(fid,'std dev.:\n');
fprintf(fid,'(%1.4f) & (%1.4f) & (%1.4f) & (%1.4f) & (%1.4f) & (%1.4f) & (%1.4f) & (%1.4f)\n',sIml
fprintf(fid,'  \n');
fprintf(fid,'*************************************************************\n');
fclose(fid);


sprintf('MISEexp with n=%g and q=%g',n,q)

end;
toc
quit
```

Appendix J. Percentage Point Generation for the Exponential with Exponential Censoring

```
% EECM3.m Exponential,Exponential Censoring Model

% Monte Carlo study of CvM and AD GOF test statistics for randomly
% censored data using the KME in place of the EDF.

% Expected proportion censored: q
% Composite Hypothesis:  Exponential with Exponential censoring

%  Written by Dave Reineke, 1998

for n=20:20:100

rand('seed',1);

nrep=250000;
m=0;

q=.9;

beta=2;
eta=50;
Theta=eta*(1-q)/q;
% Theta=415;

tic

while m < 250000

clear FAIL KM KME km U t

% Construct a randomly censored data set

for i=1:n
 t=eta*(-log(1-rand));
%  t=eta*(-log(1-rand)).^(1/beta);
%  t=5+(120-5)*rand;                        % Alternative distribution
  C=Theta*(-log(1-rand));
  X(i,1)=min(t,C);
  if C<t
    X(i,2)=0;
  else
    X(i,2)=1;
  end;
end;

[data,I]=sort(X(:,1));
for i=1:n
  data(i,2)=X(I(i),2);
end;
```

```
for i=1:n
  km(i)=((n-i)/(n-i+1)).^data(i,2);
end;
KM=cumprod(km);
KM(n)=0;

r=0;
for i=1:n
  if data(i,2)==1
    r=r+1;
    FAIL(r)=data(i,1);
    KME(r)=1-KM(i);
  end;
end;


if r>=2
m=m+1;
qobs(m)=1-r/n;
KME(r)=1;

ETAhat=sum(data(:,1))/r;  % Scale MLE for Exponential distn only

U=1-exp(-FAIL/ETAhat);

% Construct the CvM statistic

Wsum=0;
for i=2:r
Wsum=Wsum+(KME(i-1)^2)*(U(i)-U(i-1))-KME(i-1)*(U(i)^2-U(i-1)^2)+(U(i).^3-U(i-1).^3)/3;
end;
W2(m)=n*(U(1).^3)/3 + n*Wsum;

% Construct the AD statistic

if U(1)==0

    Asum=0;
    for i=3:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
    A2(m)=-n*(U(2)+log(1-U(2)))+n*Asum;

elseif U(1)>0

    Asum=0;
    for i=2:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
    A2(m)=-n*(U(1)+log(1-U(1)))+n*Asum;
end;
```

```
end;
end;

qbar=mean(qobs);

mW2=mean(W2);
sW2=std(W2);
minW2=min(W2);
medW2=prctile(W2,50);
maxW2=max(W2);

ppW2(1)=prctile(W2,75);
ppW2(2)=prctile(W2,80);
ppW2(3)=prctile(W2,85);
ppW2(4)=prctile(W2,90);
ppW2(5)=prctile(W2,95);
ppW2(6)=prctile(W2,97.5);
ppW2(7)=prctile(W2,99);
ppW2(8)=prctile(W2,99.5);
ppW2(9)=prctile(W2,99.9);

mA2=mean(A2);
sA2=std(A2);
minA2=min(A2);
medA2=prctile(A2,50);
maxA2=max(A2);

ppA2(1)=prctile(A2,75);
ppA2(2)=prctile(A2,80);
ppA2(3)=prctile(A2,85);
ppA2(4)=prctile(A2,90);
ppA2(5)=prctile(A2,95);
ppA2(6)=prctile(A2,97.5);
ppA2(7)=prctile(A2,99);
ppA2(8)=prctile(A2,99.5);
ppA2(9)=prctile(A2,99.9);


fid=fopen('EECMm9.txt','a');
fprintf(fid,'Source: EECM3.m\n');
fprintf(fid,'Sample size: n = %g\n',n);
fprintf(fid,'Monte Carlo size:   N = %g\n',m);
fprintf(fid,'Expected prop. censored: q = %1.2f\n',q);
fprintf(fid,'Observed prop. censored:     qbar = %1.2f\n',qbar);
fprintf(fid,'  \n');
fprintf(fid,'Failure dist:          EXP   scale = %g\n',eta);
fprintf(fid,'Censoring distn:       EXP   scale = %g\n',Theta);
fprintf(fid,'  \n');
fprintf(fid,'**********************************************************\n');
fprintf(fid,'  \n');
```

```
fprintf(fid,'Percentage Points of the CvM statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,' 0.25  &  0.20  &  0.15  &  0.10  &  0.05  &  0.025  & 0.01  &  0.005 & 0.001\n');
fprintf(fid,'  \n');
fprintf(fid,'%1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f\n',ppW2);
fprintf(fid,'  \n');
fprintf(fid,'Minimum                min = %1.6f\n',minW2);
fprintf(fid,'Median                              med = %1.6f\n',medW2);
fprintf(fid,'Maximum                             max = %1.6f\n',maxW2);
fprintf(fid,'Mean  E[W2] = %1.6f\n',mW2);
fprintf(fid,'Standard Deviation sd[W2] = %1.6f\n',sW2);
fprintf(fid,'  \n');
fprintf(fid,'  \n');
fprintf(fid,'Percentage Points of the AD statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,' 0.25  &  0.20  &  0.15  &  0.10  &  0.05  &  0.025  & 0.01  &  0.005 & 0.001\n');
fprintf(fid,'  \n');
fprintf(fid,'%1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f\n',ppA2);
fprintf(fid,'  \n');
fprintf(fid,'Minimum                min = %1.6f\n',minA2);
fprintf(fid,'Median                              med = %1.6f\n',medA2);
fprintf(fid,'Maximum                             max = %1.6f\n',maxA2);
fprintf(fid,'Mean  E[A2] = %1.6f\n',mA2);
fprintf(fid,'Standard Deviation sd[A2] = %1.6f\n',sA2);
fprintf(fid,'***********************************************************\n');
fprintf(fid,'  \n');
fprintf(fid,'  \n');
fclose(fid);

sprintf('Output: EECMm9.txt with n=%g and q=%g',n,q)

toc
end;
quit
```

Appendix K. Percenatge Point Generation for the Weibull with Exponential Censoring

```
% W2ECM1.m

% Monte Carlo study of the CvM and AD GOF test statistics for randomly
% censored data using the KME in place of the EDF.

% Expected proportion censored: q
% Composite Hypothesis:  Weibull with Exponential censoring

%  Written by Dave Reineke, 1998

for n=20:20:100

rand('seed',1);

nrep=250000;

m=0;

kappa=2;
eta=50;
loc=20;
Theta=80.5;
q=.40;

tic

while m < nrep

clear FAIL KM KME KMt km U t

% Construct a randomly censored data set

for i=1:n
  T=eta*(-log(1-rand)).^(1/kappa) + loc;
%  T=5+(120-5)*rand;                        % Alternative distribution
  C=Theta*(-log(1-rand)) + loc;
  X(i,1)=min(T,C);
  if C<T
    X(i,2)=0;
  else
    X(i,2)=1;
  end;
end;

[data,I]=sort(X(:,1));
for i=1:n
  data(i,2)=X(I(i),2);
end;

for i=1:n
```

```
  km(i)=((n-i)/(n-i+1)).^data(i,2);
end;
KM=cumprod(km);
KM(n)=0;

r=0;
for i=1:n
  if data(i,2)==1
    r=r+1;
    FAIL(r)=data(i,1);
    KME(r)=1-KM(i);
  end;
end;

if r>=2
m=m+1;
qobs(m)=1-r/n;
KME(r)=1;


SORTDATA=data;
% *************Find MLEs for location & scale parameters*****************
% *********************Assume shape is known******************************

% mdloc.m Finds the minimum distance estimate (MDE) of the location
%    parameter of a 3 parameter Weibull distribution assuming
% known shape.

% Written by Dave Reineke

% SORTDATA is a n by 2 matrix of failure and withdrawal times.
%       FAIL is the set of failure times only.
% KME is the Kaplan-Meier estimator of the distribution function.

GAMhat1=.999*data(1,1);
ETAhat1=(sum((data(:,1)-GAMhat1).^kappa)/r).^(1/kappa);

t=1-exp(-((FAIL-GAMhat1)./ETAhat1).^kappa);

% ***************************************************************************
% MDE using the Anderson-Darling statistic (KME vs MLE)

diff=1;

%  Golden Search algorithm

alf=2/(1+sqrt(5));

lt=0;
rt=min(FAIL);
```

```
c2=0;
while diff > .00000001
  c2=c2+1;
  if c2 == 1000
    c2
    GAMhatA=.999*data(1,1);
    break;
  end;
  x1=lt + (1-alf)*(rt-lt);
  x2=lt + alf*(rt-lt);

  t=1-exp(-((FAIL-x1)./ETAhat1).^kappa);
  if t(1)==0
    if r>2
      A1md=-(KME(2)^2)*log(t(2))-(1-(KME(2)-1)^2)*log(1-t(2));
      for i=3:r
        A1md=A1md +   (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md1=-n+n*A1md;
    elseif r==2
      A2md1=-n+n*(-log(t(2)) - log(1-t(2)));
    end;
  elseif t(1)>0
    A1md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
    for i=2:r
      A1md=A1md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
    end;
    A2md1=-n+n*A1md;
  end;

  t=1-exp(-((FAIL-x2)./ETAhat1).^kappa);
  if t(1)==0
    if r>2
      A2md=-(KME(2)^2)*log(t(2))-(1-(KME(2)-1)^2)*log(1-t(2));
      for i=3:r
        A2md=A2md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md2=-n+n*A2md;
    elseif r==2
      A2md2=-n+n*(-log(t(2)) - log(1-t(2)));
    end;
  elseif t(1)>0
    A2md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
    for i=2:r
      A2md=A2md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
    end;
    A2md2=-n+n*A2md;
  end;

  if A2md1 < A2md2
    rt=x2;
```

```
    GAMhatA=x1;
  else
    lt=x1;
    GAMhatA=x2;
  end;

  diff=abs(A2md1-A2md2);

end;

ETAhatA=(sum((data(:,1)-GAMhatA).^kappa)/r).^(1/kappa);

% ***************************************************************************
% Analysis for AD-based MDE

U=1-exp(-((FAIL-GAMhatA)/ETAhatA).^kappa);


% Construct the CvM statistic

Wsum=0;
for i=2:r
Wsum=Wsum+(KME(i-1)^2)*(U(i)-U(i-1))-KME(i-1)*(U(i)^2-U(i-1)^2)+(U(i).^3-U(i-1).^3)/3;
end;
W2(m)=n*(U(1).^3)/3 + n*Wsum;


% Construct the AD statistic

if U(1)==0

    Asum=0;
    for i=3:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
    A2(m)=-n*(U(2)+log(1-U(2)))+n*Asum;

elseif U(1)>0

    Asum=0;
    for i=2:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
    A2(m)=-n*(U(1)+log(1-U(1)))+n*Asum;
end;

end;
end;

qbar=mean(qobs);
```

```
mW2=mean(W2);
sW2=std(W2);
minW2=min(W2);
medW2=prctile(W2,50);
maxW2=max(W2);

ppW2(1)=prctile(W2,75);
ppW2(2)=prctile(W2,80);
ppW2(3)=prctile(W2,85);
ppW2(4)=prctile(W2,90);
ppW2(5)=prctile(W2,95);
ppW2(6)=prctile(W2,97.5);
ppW2(7)=prctile(W2,99);
ppW2(8)=prctile(W2,99.5);
ppW2(9)=prctile(W2,99.9);

mA2=mean(A2);
sA2=std(A2);
minA2=min(A2);
medA2=prctile(A2,50);
maxA2=max(A2);

ppA2(1)=prctile(A2,75);
ppA2(2)=prctile(A2,80);
ppA2(3)=prctile(A2,85);
ppA2(4)=prctile(A2,90);
ppA2(5)=prctile(A2,95);
ppA2(6)=prctile(A2,97.5);
ppA2(7)=prctile(A2,99);
ppA2(8)=prctile(A2,99.5);
ppA2(9)=prctile(A2,99.9);

fid=fopen('W2ECM4.txt','a');
fprintf(fid,'Source: W2ECM1.m\n');
fprintf(fid,'Sample size: n = %g\n',n);
fprintf(fid,'Monte Carlo size:   N = %g\n',m);
fprintf(fid,'Expected prop. censored: q = %1.2f\n',q);
fprintf(fid,'Observed prop. censored:qbar = %1.2f\n',qbar);
fprintf(fid,'  \n');
fprintf(fid,'Failure dist:  WEIBULL  loc= %g,scale= %g. shape= %g\n',loc,eta,kappa);
fprintf(fid,'Censoring distn: EXP   scale = %g\n',Theta);
fprintf(fid,'  \n');
fprintf(fid,'*******************************************************\n');
fprintf(fid,'  \n');
fprintf(fid,'Percentage Points of the CvM statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,' 0.25  &  0.20  &  0.15  &  0.10  &  0.05  &  0.025  & 0.01  &  0.005 & 0.001\n');
fprintf(fid,'  \n');
fprintf(fid,'%1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f\n',ppW2);
fprintf(fid,'  \n');
fprintf(fid,'Minimum              min = %1.6f\n',minW2);
```

```
fprintf(fid,'Median                                   med = %1.6f\n',medW2);
fprintf(fid,'Maximum                                  max = %1.6f\n',maxW2);
fprintf(fid,'Mean  E[W2] = %1.6f\n',mW2);
fprintf(fid,'Standard Deviation sd[W2] = %1.6f\n',sW2);
fprintf(fid,'  \n');
fprintf(fid,'  \n');
fprintf(fid,'Percentage Points of the AD statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,' 0.25  &  0.20  &  0.15  &  0.10  &  0.05  &  0.025  & 0.01  &  0.005 & 0.001\n');
fprintf(fid,'  \n');
fprintf(fid,'%1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f & %1.4f\n',ppA2);
fprintf(fid,'  \n');
fprintf(fid,'Minimum                 min = %1.6f\n',minA2);
fprintf(fid,'Median                                   med = %1.6f\n',medA2);
fprintf(fid,'Maximum                                  max = %1.6f\n',maxA2);
fprintf(fid,'Mean  E[A2] = %1.6f\n',mA2);
fprintf(fid,'Standard Deviation sd[A2] = %1.6f\n',sA2);
fprintf(fid,'*******************************************************\n');
fprintf(fid,'  \n');
fprintf(fid,'  \n');
fclose(fid);

sprintf('W2ECM1.m with n=%g, kappa=%g, and q=%g',n,kappa,q)

toc
end;
quit
```

```
% POWexp1.m Exponential,Exponential Censoring Model

% Monte Carlo POWER study of CvM and AD GOF test statistics for randomly
% censored data using the KME in place of the EDF.

% Expected proportion censored: q
% Composite Hypothesis:  Exponential with Exponential censoring

%  Written by Dave Reineke, 1999

h=0;
rejW=zeros(10,4);
rejA=zeros(10,4);

% The following matrices represent percentage points for the CvM and AD
% modified test stats: rows correspond to samples sizes 20:20:200 and
% columns correspond to alpha levels .10, .05, .025, & .01.

PPq1

rand('seed',2);

nrep=1000;

beta=2;
eta=2;
q=.1;
Theta=67.34;

for n=20:20:200
h=h+1;
m=0;

tic

while m < 1000

clear FAIL KM KME km U t

% Construct a randomly censored data set

for i=1:n
% t=eta*(-log(1-rand)); % Lifetime distribution (Exp)
% t=eta*(-log(1-rand)).^(1/beta);        % Weibull Alternative
% t=exp(normrnd(0,1)); % Lognormal Alternative
% t=gamrnd(beta,eta); % Gamma(chi-sq.) Alt.
  t=100*rand; % Uniform(0,100) Alt.
  C=Theta*(-log(1-rand));  % Censoring Distribution (Exp)
  X(i,1)=min(t,C);
  if C<t
    X(i,2)=0;
```

```
  else
    X(i,2)=1;
  end;
end;

[data,I]=sort(X(:,1));
for i=1:n
  data(i,2)=X(I(i),2);
end;

for i=1:n
  km(i)=((n-i)/(n-i+1)).^data(i,2);
end;
KM=cumprod(km);
KM(n)=0;

r=0;
for i=1:n
  if data(i,2)==1
    r=r+1;
    FAIL(r)=data(i,1);
    KME(r)=1-KM(i);
  end;
end;


if r>=2
m=m+1;
qobs(m)=1-r/n;
KME(r)=1;

ETAhat=(sum(data(:,1))/r);  % Scale MLE for Exponential distn only

U=1-exp(-FAIL/ETAhat);

% Construct the CvM statistic

Wsum=0;
for i=2:r
Wsum=Wsum+(KME(i-1)^2)*(U(i)-U(i-1))-KME(i-1)*(U(i)^2-U(i-1)^2)+(U(i).^3-U(i-1).^3)/3;
end;
W2=n*(U(1).^3)/3 + n*Wsum;

% Construct the AD statistic

if U(1)==0

    Asum=0;
    for i=3:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
```

```
    A2=-n*(U(2)+log(1-U(2)))+n*Asum;

elseif U(1)>0

    Asum=0;
    for i=2:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
    A2=-n*(U(1)+log(1-U(1)))+n*Asum;
end;

%  Tally the number of test statistics over given percentage points

for i=1:4
  if W2 > ppW(h,i)
    rejW(h,i)=rejW(h,i)+1;
  end;
  if A2 > ppA(h,i)
    rejA(h,i)=rejA(h,i)+1;
  end;
end;

end;
end;

qbar=mean(qobs);

powW=rejW/m;

powA=rejA/m;

fid=fopen('POWexp1.txt','a');
fprintf(fid,'Source: POWexp1.m\n');
fprintf(fid,'Sample size: n = %g\n',n);
fprintf(fid,'Monte Carlo size:   N = %g\n',m);
fprintf(fid,'Expected prop. censored: q = %1.2f\n',q);
fprintf(fid,'Observed prop. censored:     qbar = %1.2f\n',qbar);
fprintf(fid,'  \n');
%fprintf(fid,'Failure distn (Alt):  WEIBULL shape = %g,   scale =%g\n',beta,eta);
fprintf(fid,'Alternative distn:    Uniform (0,100)\n');
fprintf(fid,'Censoring distn:      EXP    scale = %g\n',Theta);
fprintf(fid,'  \n');
fprintf(fid,'Hypothesized Distribution:  Exponential\n');
fprintf(fid,'**********************************************************\n');
fprintf(fid,'  \n');
fprintf(fid,'Estimated power of the CvM statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,' 0.10  &  0.05  &  0.025  &  0.01\n');
fprintf(fid,'  \n');
fprintf(fid,'%1.4f & %1.4f & %1.4f & %1.4f \n',powW(h,:)');
fprintf(fid,'  \n');
```

```
fprintf(fid,'  \n');
fprintf(fid,'Estimated power of the AD statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,' 0.10  &  0.05  &  0.025  &  0.01\n');
fprintf(fid,'  \n');
fprintf(fid,'%1.4f & %1.4f & %1.4f & %1.4f \n',powA(h,:)');
fprintf(fid,'  \n');
fprintf(fid,'*********************************************************\n');
fprintf(fid,'  \n');
fclose(fid);

sprintf('POWexp1.m with n=%g and q=%g',n,q)

toc
end;
quit

% POWcrd1.m  Power study for the simultaneous tests of crude lifetimes and
% semi-parametric tests of fit with crude lifetimes for the
% exponential distribution with exponential censoring.

%  Written by Dave Reineke, May 1999

h=0;
rejW1=zeros(5,3);
rejA1=zeros(5,3);
rejW2=zeros(5,3);
rejA2=zeros(5,3);

% The following matrices represent percentage points for the CvM and AD
% modified test stats: rows correspond to samples sizes 20:20:100 and
% columns correspond to alpha levels .10, .05, .025, & .01

ppW=[.175 .222 .271 .338];
ppA=[1.062 1.321 1.591 1.959];


rand('seed',2);

nrep=1000;

beta=2;
eta=2;
q=.2;
Theta=17;

for n=20:20:100
h=h+1;
m=0;

tic
```

```
while m < nrep

clear FAIL z CENS

% Construct a randomly censored data set

for i=1:n
% t=eta*(-log(1-rand)); % Lifetime distribution (Exp)
% t=eta*(-log(1-rand)).^(1/beta);       % Weibull Alternative
% t=exp(normrnd(0,1)); % Lognormal Alternative
 t=gamrnd(beta,eta); % Gamma(chi-sq.) Alt.
  C=Theta*(-log(1-rand));  % Censoring Distribution (Exp)
  X(i,1)=min(t,C);
  if C<t
    X(i,2)=0;
  else
    X(i,2)=1;
  end;
end;

[data,I]=sort(X(:,1));
for i=1:n
  data(i,2)=X(I(i),2);
end;


% Separate Crude Failure and Censoring Times

r=0;
K=0;
for i=1:n
  if data(i,2)==1
    r=r+1;
    FAIL(r)=data(i,1);
  elseif data(i,2)==0
    K=K+1;
    CENS(K)=data(i,1);
  end;
end;

if r>=2
if K>=2
m=m+1;
qobs(m)=1-r/n;

% Find Scale Parameter Estimates for Each "Exponential" crude lifetime distribution

L1=mean(FAIL);
L2=mean(CENS);
```

```
% Conduct CvM Tests for the Exponential for the FAIL set and CENS set

z1=1-exp(-FAIL/L1);

W=0;
for i=1:r
  W=W + (z1(i) - (i-.5)/r)^2;
end;
W=W + 1/(12*r);
W1=W*(1 + 0.16/r); % CvM stat for FAIL set

z2=1-exp(-CENS/L2);

W=0;
for i=1:K
  W=W + (z2(i) - (i-.5)/K)^2;
end;
W=W + 1/(12*K);
W2=W*(1 + 0.16/K); % CvM stat for CENS set

% Conduct AD Tests for the Exponential for the FAIL set and CENS set

A=0;
for i=1:r
  A=A + (2*i-1)*(log(z1(i)) + log(1 - z1(r+1-i)));
end;
A= -r - (1/r)*A;
A1= A*(1 + 0.6/r); % AD stat for FAIL set

A=0;
for i=1:K
  A=A + (2*i-1)*(log(z2(i)) + log(1 - z2(K+1-i)));
end;
A= -K - (1/K)*A;
A2= A*(1 + 0.6/K); % AD stat for CENS set

%  Tally the number of test statistics over given percentage points

%  For the Simultaneous Crude Life Tests (Reject if EITHER test rejects)

for i=1:3
  rejW1(h,i)=rejW1(h,i)+1;
  if W1 < ppW(i+1)
    if W2 < ppW(i+1)
      rejW1(h,i)=rejW1(h,i)-1;
    end;
  end;

  rejA1(h,i)=rejA1(h,i)+1;
  if A1 < ppA(i+1)
    if A2 < ppA(i+1)
```

```
      rejA1(h,i)=rejA1(h,i)-1;
    end;
  end;
end;

% For the Semi-Parametric Crude Life Tests

for i=1:3
  if W1 > ppW(i)
    rejW2(h,i)=rejW2(h,i)+1;
  end;
  if A1 > ppA(i)
    rejA2(h,i)=rejA2(h,i)+1;
  end;
end;

end;
end;
end;

qbar=mean(qobs);

powW1=rejW1/m;
powA1=rejA1/m;

powW2=rejW2/m;
powA2=rejA2/m;

fid=fopen('POWcrd1.txt','a');
fprintf(fid,'Source: POWcrd1.m\n');
fprintf(fid,'Sample size: n = %g\n',n);
fprintf(fid,'Monte Carlo size:   N = %g\n',m);
fprintf(fid,'Expected prop. censored: q = %1.2f\n',q);
fprintf(fid,'Observed prop. censored:      qbar = %1.2f\n',qbar);
fprintf(fid,'  \n');
%fprintf(fid,'Failure distn:         EXP   scale = %g\n',eta);
%fprintf(fid,'Failure distn (Alt):  WEIBULL shape = %g,  scale =%g\n',beta,eta);
fprintf(fid,'Alternative distn:    Gamma shape=%g, scale=%g\n',beta,eta);
fprintf(fid,'Censoring distn:      EXP   scale = %g\n',Theta);
fprintf(fid,'  \n');
fprintf(fid,'Hypothesized Distribution:  Exponential\n');
fprintf(fid,'*********************************************************\n');
fprintf(fid,'SIMULTANEOUS CRUDE LIFE TEST  \n');
fprintf(fid,'Estimated power of the CvM and AD statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,'   0.10  &  0.05  &  0.025  \n');
fprintf(fid,'  \n');
fprintf(fid,'CvM %1.4f & %1.4f & %1.4f \n',powW1(h,:)');
fprintf(fid,'AD %1.4f & %1.4f & %1.4f \n',powA1(h,:)');
fprintf(fid,'  \n');
fprintf(fid,'  \n');
```

```
fprintf(fid,'SEMI-PARAMETRIC CRUDE LIFE TEST  \n');
fprintf(fid,'Estimated power of the CvM statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,'  0.10  &  0.05  &  0.025  \n');
fprintf(fid,'  \n');
fprintf(fid,'CvM %1.4f & %1.4f & %1.4f \n',powW2(h,:)');
fprintf(fid,'AD %1.4f & %1.4f & %1.4f \n',powA2(h,:)');
fprintf(fid,'**********************************************************\n');
fprintf(fid,'  \n');
fclose(fid);

sprintf('POWcrd1.m with n=%g and q=%g',n,q)

toc

end;
```

```
% POWw2exp1.m Exponential,Exponential Censoring Model

% Monte Carlo POWER study of CvM and AD GOF test statistics for randomly
% censored data using the KME in place of the EDF.

% Expected proportion censored: q
% Composite Hypothesis:  Exponential with Exponential censoring

%  Written by Dave Reineke, 1999

h=0;
rejW=zeros(5,4);
rejA=zeros(5,4);

tic

% The following matrices represent percentage points for the CvM and AD
% modified test stats: rows correspond to samples sizes 20:20:100 and
% columns correspond to alpha levels .10, .05, .025, & .01.

ppW=[0.171 0.208 0.245 0.294
     0.168 0.206 0.243 0.293
     0.167 0.204 0.241 0.293
     0.165 0.202 0.239 0.288
     0.164 0.201 0.238 0.288];

ppA=[1.091 1.301 1.504 1.792
     1.076 1.286 1.496 1.768
     1.064 1.270 1.476 1.758
     1.048 1.253 1.457 1.726
     1.038 1.243 1.443 1.715];

rand('seed',2);

nrep=1000;

kappa=2;
beta=2;
eta=50;
loc=20;
q=.2;
Theta=193;

for n=20:20:100
h=h+1;
m=0;

while m < nrep

clear FAIL KM KME km U t
```

```
% Construct a randomly censored data set

for i=1:n
% t=eta*(-log(1-rand))+loc; % Lifetime distribution (Exp)
 t=eta*(-log(1-rand)).^(1/beta)+loc;       % Weibull Alternative
% t=exp(normrnd(0.4,0.67))+loc; % Lognormal Alternative
% t=gamrnd(beta,eta)+loc; % Gamma(chi-sq.) Alt.
  C=Theta*(-log(1-rand))+loc;  % Censoring Distribution (Exp)
  X(i,1)=min(t,C);
  if C<t
    X(i,2)=0;
  else
    X(i,2)=1;
  end;
end;

[data,I]=sort(X(:,1));
for i=1:n
  data(i,2)=X(I(i),2);
end;

for i=1:n
  km(i)=((n-i)/(n-i+1)).^data(i,2);
end;
KM=cumprod(km);
KM(n)=0;

r=0;
for i=1:n
  if data(i,2)==1
    r=r+1;
    FAIL(r)=data(i,1);
    KME(r)=1-KM(i);
  end;
end;


if r>=2
m=m+1;
qobs(m)=1-r/n;
KME(r)=1;

%  MD/ML estimation of location and scale parameters

SORTDATA=data;
% *************Find MLEs for location & scale parameters*****************
% *********************Assume shape is known*****************************

% mdloc.m Finds the minimum distance estimate (MDE) of the location
%    parameter of a 3 parameter Weibull distribution assuming
% known shape.
```

```
% Written by Dave Reineke

% SORTDATA is a n by 2 matrix of failure and withdrawal times.
%       FAIL is the set of failure times only.
% KME is the Kaplan-Meier estimator of the distribution function.

GAMhat1=.999*data(1,1);
ETAhat1=(sum((data(:,1)-GAMhat1).^kappa)/r).^(1/kappa);

t=1-exp(-((FAIL-GAMhat1)./ETAhat1).^kappa);

% **************************************************************************
% MDE using the Anderson-Darling statistic (KME vs MLE)

diff=1;

%  Golden Search algorithm

alf=2/(1+sqrt(5));

lt=0;
rt=min(FAIL);

c2=0;
while diff > .00000001
  c2=c2+1;
  if c2 == 1000
    c2
    GAMhatA=.999*data(1,1);
    break;
  end;
  x1=lt + (1-alf)*(rt-lt);
  x2=lt + alf*(rt-lt);

  t=1-exp(-((FAIL-x1)./ETAhat1).^kappa);
  if t(1)==0
    if r>2
      A1md=-(KME(2)^2)*log(t(2))-(1-(KME(2)-1)^2)*log(1-t(2));
      for i=3:r
        A1md=A1md +   (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md1=-n+n*A1md;
    elseif r==2
      A2md1=-n+n*(-log(t(2)) - log(1-t(2)));
    end;
  elseif t(1)>0
    A1md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
    for i=2:r
      A1md=A1md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
    end;
```

```
    A2md1=-n+n*A1md;
  end;

  t=1-exp(-((FAIL-x2)./ETAhat1).^kappa);
  if t(1)==0
    if r>2
      A2md=-(KME(2)^2)*log(t(2))-(1-(KME(2)-1)^2)*log(1-t(2));
      for i=3:r
        A2md=A2md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
      end;
      A2md2=-n+n*A2md;
    elseif r==2
      A2md2=-n+n*(-log(t(2)) - log(1-t(2)));
    end;
  elseif t(1)>0
    A2md=-(KME(1)^2)*log(t(1))-(1-(KME(1)-1)^2)*log(1-t(1));
    for i=2:r
      A2md=A2md + (KME(i-1)^2-KME(i)^2)*log(t(i))-((KME(i-1)-1)^2-(KME(i)-1)^2)*log(1-t(i));
    end;
    A2md2=-n+n*A2md;
  end;

  if A2md1 < A2md2
    rt=x2;
    GAMhat=x1;
  else
    lt=x1;
    GAMhat=x2;
  end;

  diff=abs(A2md1-A2md2);

end;

ETAhat=(sum((data(:,1)-GAMhat).^kappa)/r).^(1/kappa);

% ***************************************************************************


U=1-exp(-((FAIL-GAMhat)/ETAhat).^kappa);

% Construct the CvM statistic

Wsum=0;
for i=2:r
Wsum=Wsum+(KME(i-1)^2)*(U(i)-U(i-1))-KME(i-1)*(U(i)^2-U(i-1)^2)+(U(i).^3-U(i-1).^3)/3;
end;
W2=n*(U(1).^3)/3 + n*Wsum;

% Construct the AD statistic
```

```
if U(1)==0

    Asum=0;
    for i=3:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
    A2=-n*(U(2)+log(1-U(2)))+n*Asum;

elseif U(1)>0

    Asum=0;
    for i=2:r
      Asum=Asum+(KME(i-1).^2)*(log(U(i))-log(U(i-1)))-((KME(i-1)-1).^2)*(log(1-U(i))-log(1-U(i-1))
    end;
    A2=-n*(U(1)+log(1-U(1)))+n*Asum;
end;

%  Tally the number of test statistics over given percentage points

for i=1:4
  if W2 > ppW(h,i)
    rejW(h,i)=rejW(h,i)+1;
  end;
  if A2 > ppA(h,i)
    rejA(h,i)=rejA(h,i)+1;
  end;
end;

end;
end;

qbar=mean(qobs);

powW=rejW/m;

powA=rejA/m;

fid=fopen('POWw2exp2.txt','a');
fprintf(fid,'Source: POWw2exp1.m\n');
fprintf(fid,'Sample size: n = %g\n',n);
fprintf(fid,'Monte Carlo size:   N = %g\n',m);
fprintf(fid,'Expected prop. censored: q = %1.2f\n',q);
fprintf(fid,'Observed prop. censored:     qbar = %1.2f\n',qbar);
fprintf(fid,'  \n');
fprintf(fid,'Failure distn (Alt):  WEIBULL shape = %g,  scale =%g\n',beta,eta);
%fprintf(fid,'Alternative distn:    Lognormal from N(.4,.67)\n');
fprintf(fid,'Censoring distn:      EXP   scale = %g\n',Theta);
fprintf(fid,'  \n');
fprintf(fid,'Hypothesized Distribution:  Exponential\n');
fprintf(fid,'*********************************************************\n');
fprintf(fid,'  \n');
```

```
fprintf(fid,'Estimated power of the CvM statistic: Composite Ho\n');
fprintf(fid,'  \n');
fprintf(fid,'            0.10  &  0.05  &  0.025  &  0.01\n');
fprintf(fid,'  \n');
fprintf(fid,'CvM Stat:  %1.3f & %1.3f & %1.3f & %1.3f \n',powW(h,:)');
fprintf(fid,'AD Stat:   %1.3f & %1.3f & %1.3f & %1.3f \n',powA(h,:)');
fprintf(fid,'  \n');
fprintf(fid,'********************************************************\n');
fprintf(fid,'  \n');
fclose(fid);

sprintf('POWw2exp1.m with n=%g and q=%g',n,q)

end;
toc
quit
```

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13. ABSTRACT *(Maximum 200 words)*
A new continuous distribution function estimator for randomly censored data is developed, discussed, and compared to existing estimators. Minimum distance estimation is shown to be effective in estimating Weibull location parameters when random censoring is present. A method of estimating all 3 parameters of the 3-parameter Weibull distribution using a combination of minimum distance and maximum likelihood is also given. Cramér-von Mises and Anderson-Darling goodness-of-fit test statistics are modified to measure the discrepancy between the maximum likelihood estimate and the Kaplan-Meier product-limit estimate of the distribution function of the random variable of interest. These modified test statistics are used to construct goodness-of-fit tests for the exponential, Weibull (shape 2), and Weibull (shape 3.5) distributions when the censoring distribution is assumed to be exponential. Percentage points are obtained via Monte Carlo simulation. More generally, elements of competing risks theory are used to build goodness-of-fit tests using crude lifetimes. For tests based on crude lifetimes, the assumption of an exponentially distributed censoring variable and special estimation techniques are no longer required. Further, complete sample goodness-of-fit techniques may be used, bringing much more flexibility to goodness-of-fit testing when samples are randomly right-censored.

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